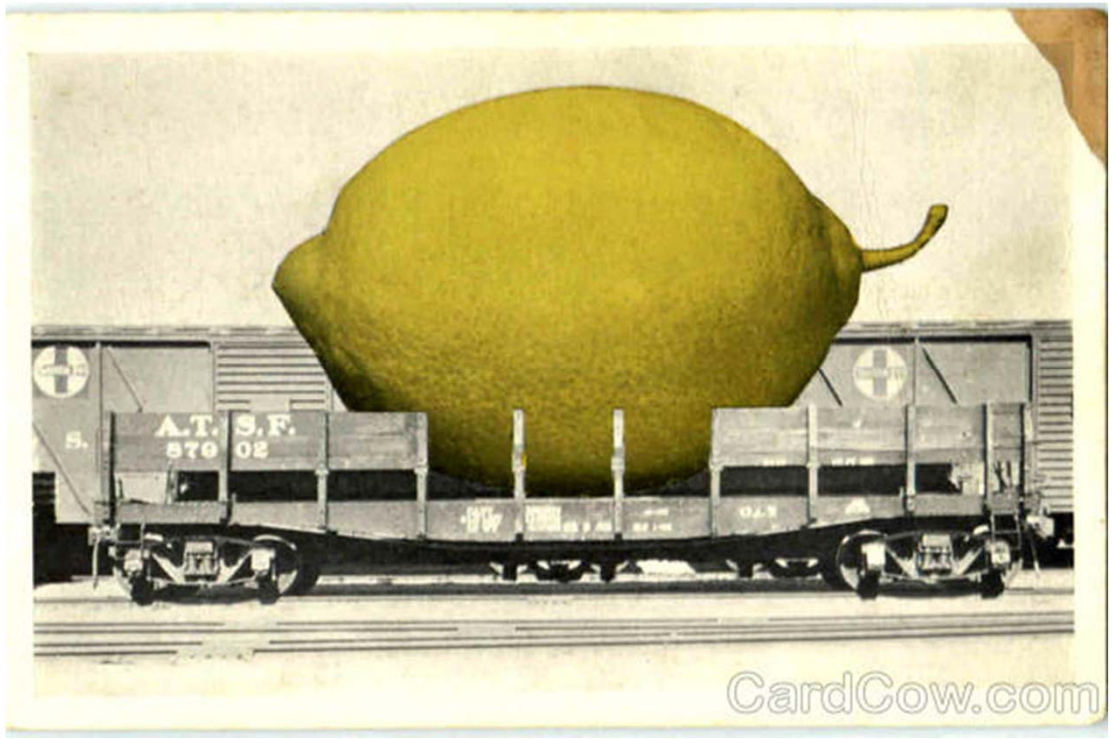
Importance Sampling



What is Importance Sampling ?

- A simulation technique
- Used when we are interested in rare events
- Examples:
  - Bit Error Rate on a channel,
  - Failure probability of a reliable system

We saw some of it already

- **Q:** We simulate R = 10 000 samples and find no bit error. What can we say about the bit error rate ?

- **A:** with confidence 0.95, BER < 3.7 10^{-4}

THEOREM 2.2.4. *[37, p. 110] Assume we observe z successes out of n independent experiments. A confidence interval at level γ for the success probability p is $[L(z); U(z)]$ with*

$$\begin{cases} L(0) = 0 \\ L(z) = \phi_{N,z-1}\left(\frac{1+\gamma}{2}\right), \ z = 1, ..., n \\ U(z) = 1 - L(n-z) \end{cases} \tag{2.26}$$

where $\phi_{n,z}(\alpha)$ is defined for $n = 2, 3, ...,$ $z \in \{0, 1, ..., n\}$ and $\alpha \in (0; 1)$ by

$$\begin{cases} \phi_{n,z}(\alpha) = \frac{n_1 f}{n_2 + n_1 f} \\ n_1 = 2(z+1), \ n_2 = 2(n-z), \ 1 - \alpha = F_{n_1,n_2}(f) \end{cases} \tag{2.27}$$

($F_{n_1,n_2}()$ is the CDF of the Fisher distribution with n_1, n_2 degrees of freedom). In particular, the confidence interval for p when we observe $z = 0$ successes is $[0; p_0(n)]$ with

$$p_0(n) = 1 - \left(\frac{1-\gamma}{2}\right)^{\frac{1}{n}} = \frac{1}{n}\log\left(\frac{2}{1-\gamma}\right) + o\left(\frac{1}{n}\right) \text{ for large } n \tag{2.28}$$

Whenever $z \geq 6$ and $n - z \geq 6$, the normal approximation

$$\begin{cases} L(z) \approx \frac{z}{n} - \frac{\eta}{n}\sqrt{z\left(1 - \frac{z}{n}\right)} \\ U(z) \approx \frac{z}{n} + \frac{\eta}{n}\sqrt{z\left(1 - \frac{z}{n}\right)} \end{cases} \tag{2.29}$$

can be used instead, with $N_{0,1}(\eta) = \frac{1+\gamma}{2}$.

What is the Problem ?

- Assume you can simulate a system
- You want to evaluate the probability of a rare event
- We want to say more than an answer like : $p \in [0, 3.69\ 10^{-4}]$

i.e. we want a good relative accuracy on p

- Assume proba of rare event is 10^{-6}: how many simulation runs do you need to obtain an estimate of p with 10% relative accuracy ?

What is the Problem ?

- Assume proba of rare event is 10^{-6}: how many simulation runs do you need to obtain an estimate of p with 10% relative accuracy ?

What is the Problem ?

- Assume proba of rare event is 10^{-6}: how many simulation runs do you need to obtain an estimate of p with 10% relative accuracy ?

R replications
N events

$$\hat{p} = \frac{N}{R}$$

Confidence interval $\hat{p} \pm 1.96 \frac{\sigma}{\sqrt{R}}$

$$\sigma^2 \approx \hat{p}(1-\hat{p})$$

$$\text{Relative accuracy} = \frac{1.96\,\sigma}{\sqrt{R}\,\hat{p}} = 1.96 \frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{R}} = 1.96 \sqrt{\frac{1-\hat{p}}{R\hat{p}}}$$

$$\text{Relative accuracy} = 10\% \Leftrightarrow 1.96 \sqrt{\frac{1-\hat{p}}{R\hat{p}}} = 0.1 \Leftrightarrow R \approx \frac{1.96^2}{0.1^2 p} \approx \frac{400}{p}$$

The Goal of Importance Sampling

- Obtain small probability p with good accuracy
- … while keeping R small

- In the previous example, the direct approach requires $R = 4.10^8$ runs to estimate $p \approx 10^{-6}$ with 10% accuracy

- We can do much better with Importance Sampling

The Idea of Importance Sampling

Formally, assume we simulate a random variable X in $\mathbb{R}^d$, with PDF $f_X()$. Our goal is to estimate $p = \mathbb{E}(\phi(X))$, where ϕ is the metric of interest. Frequently, $\phi(x)$ is the indicator function, equal to 1 if the value x corresponds to a failure of the system, and 0 otherwise.

We replace the original PDF $f_X()$ by another one, $f_{\hat{X}}()$, called the PDF of the *importance sampling distribution*, on the same space $\mathbb{R}^d$. We assume that

$$\text{if } f_X(x) > 0 \text{ then } f_{\hat{X}}(x) > 0$$

i.e. the support of the importance sampling distribution contains that of the original one. For x in the support of $f_X()$, define the *weighting function*

$$w(x) = \frac{f_X(x)}{f_{\hat{X}}(x)} \tag{7.15}$$

The Idea of Importance Sampling (cont'd)

- If we simulate X, how do we estimate p ?

- If we simulate $\hat{X}$ instead of X, we cannot use $E(\phi(\hat{X}))$

- But: $E\left(\phi(\hat{X})w(\hat{X})\right) = p$

  Show this !

Importance Sampling Monte Carlo

which is the fundamental equation of importance sampling. A Monte Carlo estimate of p is thus given by

$$\hat{p} = \frac{1}{R} \sum_{r=1}^{R} \phi(\hat{X}_r) w(\hat{X}_r) \tag{7.17}$$

where $\hat{X}_r$ are R independent replicates of $\hat{X}$.

Example: Bit Error Rate (BER)

EXAMPLE 7.16: BIT ERROR RATE AND EXPONENTIAL TWISTING. The Bit Error Rate on a communication channel with impulsive interferers can be expressed as [25]:

$$p = \mathbb{P}\left(X_0 + X_1 + ... + X_d > a\right) \tag{7.18}$$

where $X_0 \sim N_{0,\sigma^2}$ is thermal noise and X_j, $j = 1, ..., d$ represents impulsive interferers. The distribution of X_j is discrete, with support in $\{\pm x_{j,k}, k = 1, ..., n\} \cup \{0\}$ and:

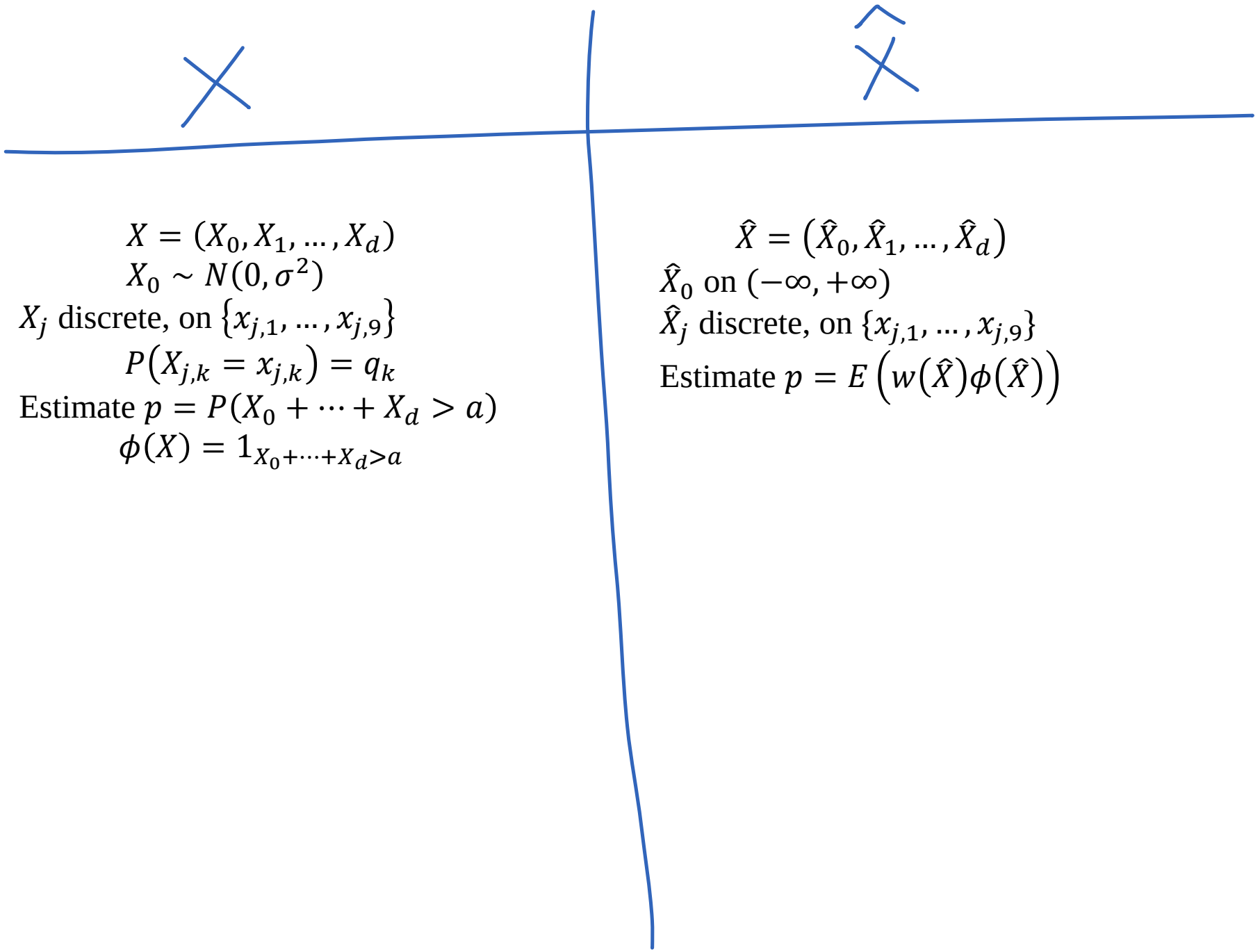
$$\begin{aligned} \mathbb{P}(X_j = \pm x_{j,k}) &= q \\ \mathbb{P}(X_j = 0) &= 1 - 2nq \end{aligned}$$

where $n = 40$, $q = \frac{1}{512}$ and the array $\{\pm x_{j,k}, k = 1, ..., n\}$ are given numerically by channel estimation (Table 7.2, for $d = 9$). The variables $X_j, j = 0, ..., d$ are independent. For large values of d, we could approximate p by a gaussian approximation, but it can easily be verified that for d of the order of 10 or less this does not hold [25].

k	j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8	j=9
1	0.4706	0.0547	0.0806	0.0944	0.4884	0.3324	0.4822	0.3794	0.2047
2	0.8429	0.0683	0.2684	0.2608	0.0630	0.1022	0.1224	0.0100	0.0282
...	...								...

A direct Monte Carlo estimation (without importance sampling) gives the following results (R is the number of Monte Carlo runs required to reach 10% accuracy with confidence 95%, as of Eq.(7.14)):

σ	a	BER estimate	R
0.1	3	$(6.45 \pm 0.6) \times 10^{-6}$	6.2×10^7



$X = (X_0, X_1, \dots, X_d)$

$X_0 \sim N(0, \sigma^2)$

X_j discrete, on $\{x_{j,1}, \dots, x_{j,9}\}$

$P(X_{j,k} = x_{j,k}) = q_k$

Estimate $p = P(X_0 + \cdots + X_d > a)$

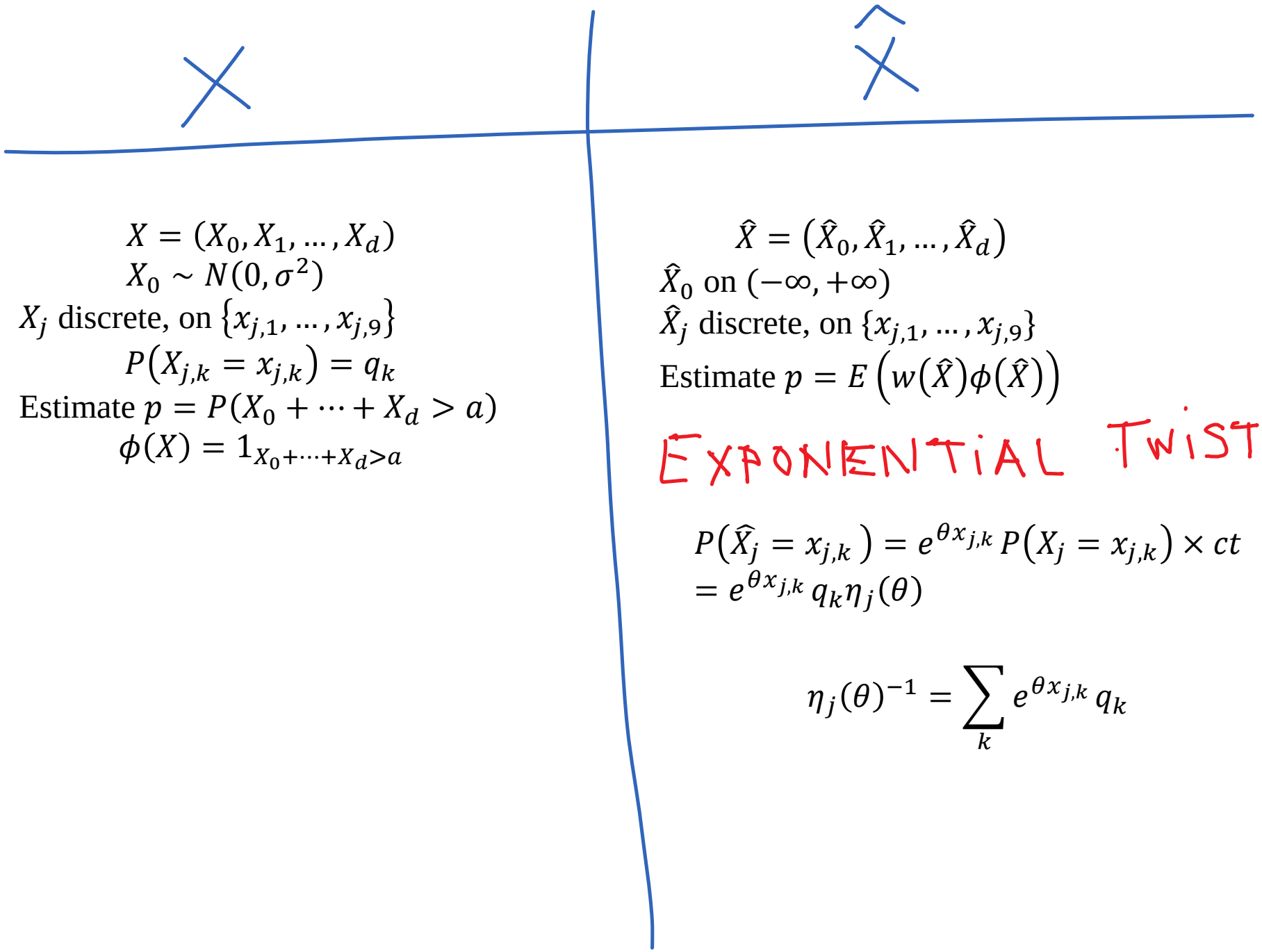
$\phi(X) = 1_{X_0 + \cdots + X_d > a}$

$\hat{X} = (\hat{X}_0, \hat{X}_1, \dots, \hat{X}_d)$

$\hat{X}_0$ on $(-\infty, +\infty)$

$\hat{X}_j$ discrete, on $\{x_{j,1}, \dots, x_{j,9}\}$

Estimate $p = E\left(w(\hat{X})\phi(\hat{X})\right)$



$X = (X_0, X_1, \dots, X_d)$

$X_0 \sim N(0, \sigma^2)$

X_j discrete, on $\{x_{j,1}, \dots, x_{j,9}\}$

$P(X_{j,k} = x_{j,k}) = q_k$

Estimate $p = P(X_0 + \cdots + X_d > a)$

$\phi(X) = 1_{X_0 + \cdots + X_d > a}$

$\hat{X} = (\hat{X}_0, \hat{X}_1, \dots, \hat{X}_d)$

$\hat{X}_0$ on $(-\infty, +\infty)$

$\hat{X}_j$ discrete, on $\{x_{j,1}, \dots, x_{j,9}\}$

Estimate $p = E\left(w(\hat{X})\phi(\hat{X})\right)$

EXPONENTIAL TWIST

$P(\hat{X}_j = x_{j,k}) = e^{\theta x_{j,k}} P(X_j = x_{j,k}) \times ct$

$= e^{\theta x_{j,k}} q_k \eta_j(\theta)$

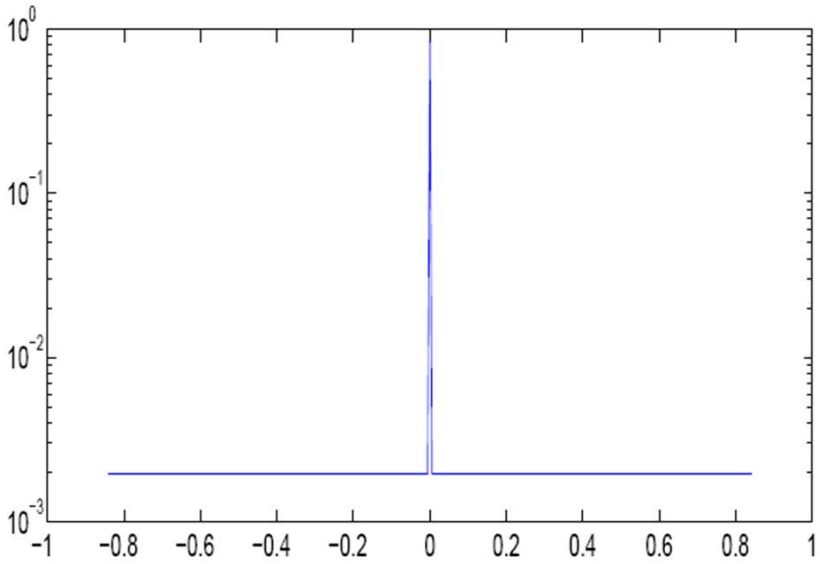
$$\eta_j(\theta)^{-1} = \sum_k e^{\theta x_{j,k}} q_k$$

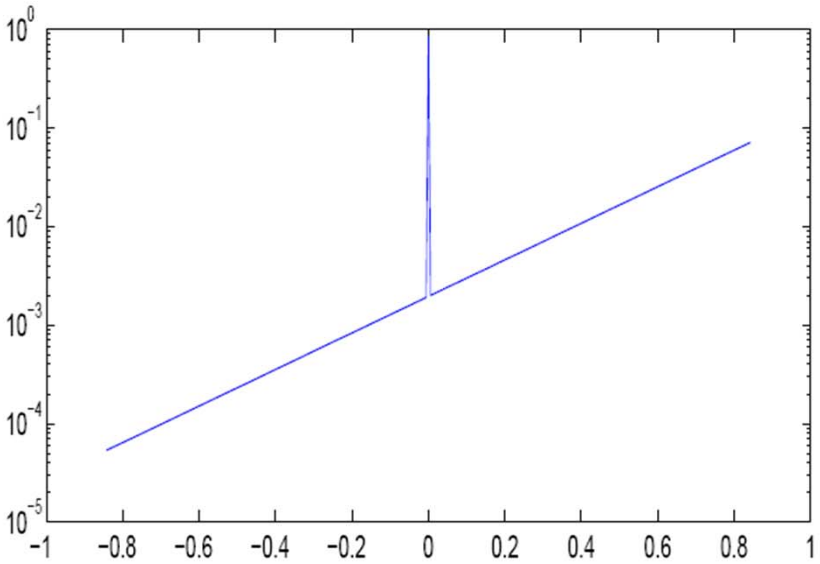
X_i

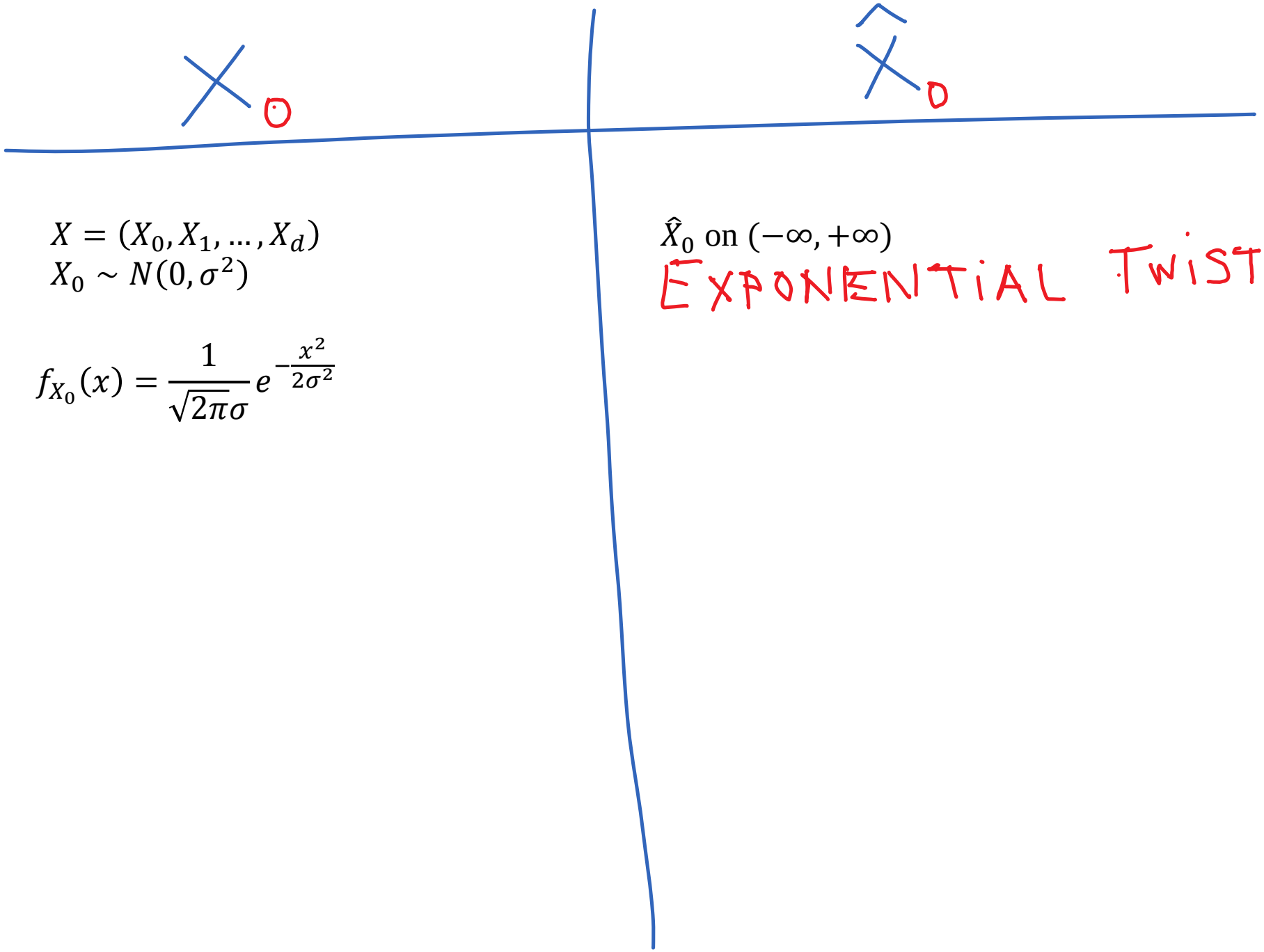
$\hat{X}_i$

PDF

PDF





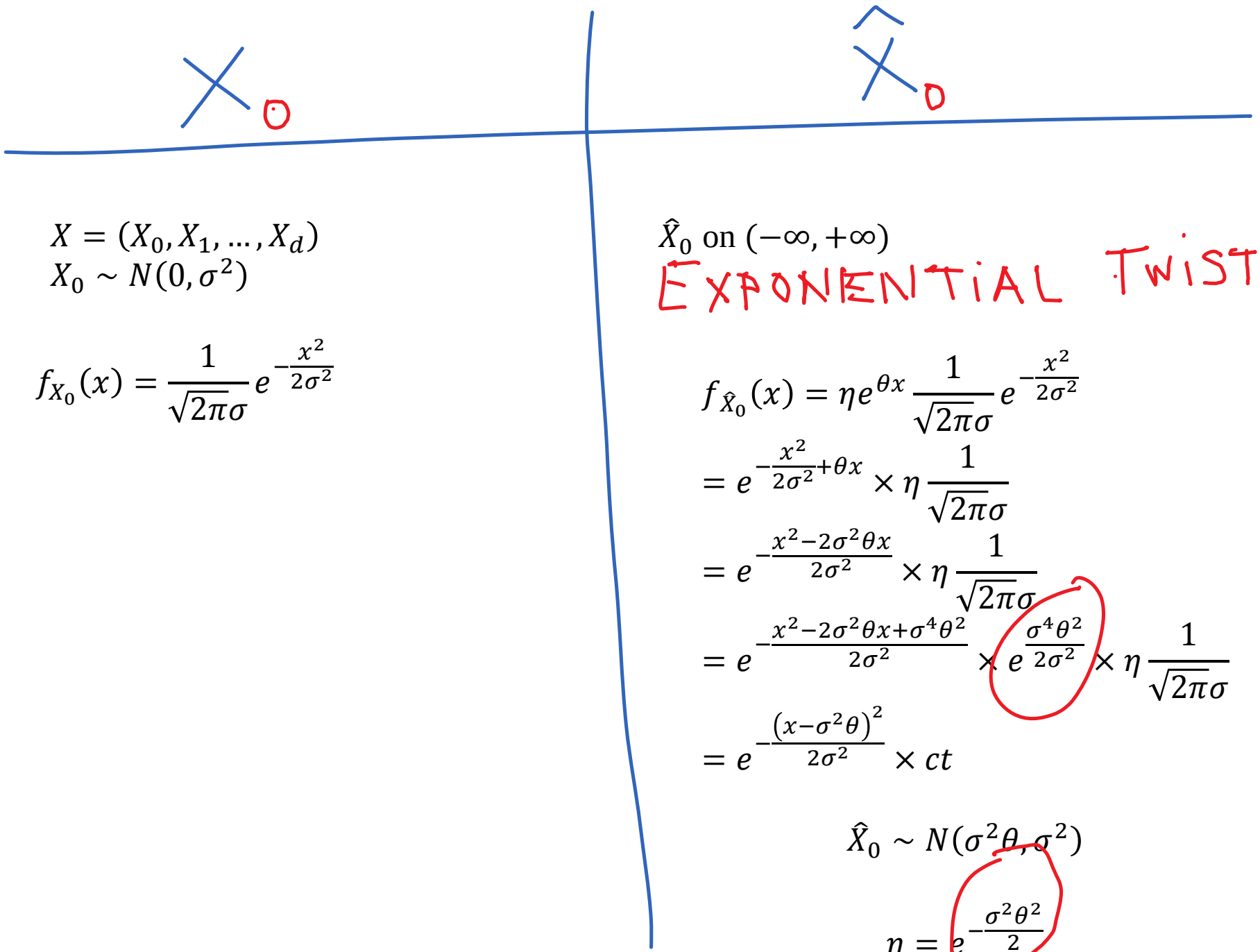


$X = (X_0, X_1, \ldots, X_d)$
$X_0 \sim N(0, \sigma^2)$

$$f_{X_0}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$\hat{X}_0$ on $(-\infty, +\infty)$

EXPONENTIAL TWIST



$X = (X_0, X_1, \ldots, X_d)$
$X_0 \sim N(0, \sigma^2)$

$$f_{X_0}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

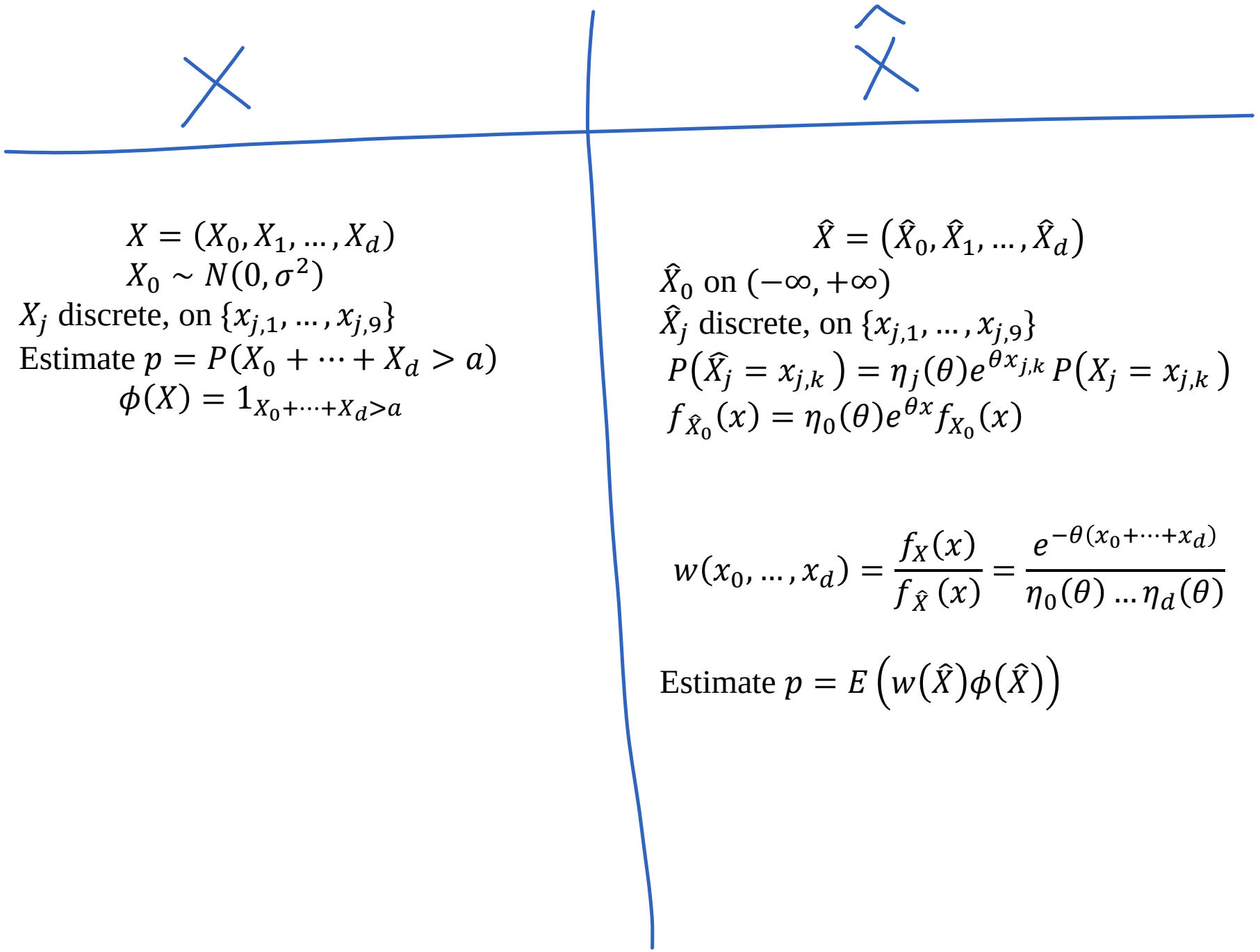
$\hat{X}_0$ on $(-\infty, +\infty)$

EXPONENTIAL TWIST

$$f_{\hat{X}_0}(x) = \eta e^{\theta x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$
$$= e^{-\frac{x^2}{2\sigma^2}+\theta x} \times \eta \frac{1}{\sqrt{2\pi}\sigma}$$
$$= e^{-\frac{x^2-2\sigma^2\theta x}{2\sigma^2}} \times \eta \frac{1}{\sqrt{2\pi}\sigma}$$
$$= e^{-\frac{x^2-2\sigma^2\theta x+\sigma^4\theta^2}{2\sigma^2}} \times e^{\frac{\sigma^4\theta^2}{2\sigma^2}} \times \eta \frac{1}{\sqrt{2\pi}\sigma}$$
$$= e^{-\frac{(x-\sigma^2\theta)^2}{2\sigma^2}} \times ct$$

$$\hat{X}_0 \sim N(\sigma^2\theta, \sigma^2)$$

$$\eta = e^{-\frac{\sigma^2\theta^2}{2}}$$



$X = (X_0, X_1, \dots, X_d)$

$X_0 \sim N(0, \sigma^2)$

X_j discrete, on $\{x_{j,1}, \dots, x_{j,9}\}$

Estimate $p = P(X_0 + \cdots + X_d > a)$

$\phi(X) = 1_{X_0 + \cdots + X_d > a}$

$\hat{X} = (\hat{X}_0, \hat{X}_1, \dots, \hat{X}_d)$

$\hat{X}_0$ on $(-\infty, +\infty)$

$\hat{X}_j$ discrete, on $\{x_{j,1}, \dots, x_{j,9}\}$

$P(\hat{X}_j = x_{j,k}) = \eta_j(\theta) e^{\theta x_{j,k}} P(X_j = x_{j,k})$

$f_{\hat{X}_0}(x) = \eta_0(\theta) e^{\theta x} f_{X_0}(x)$

$$w(x_0, \dots, x_d) = \frac{f_X(x)}{f_{\hat{X}}(x)} = \frac{e^{-\theta(x_0 + \cdots + x_d)}}{\eta_0(\theta) \dots \eta_d(\theta)}$$

Estimate $p = E\left(w(\hat{X})\phi(\hat{X})\right)$

Importance Sampling Monte Carlo

We perform R Monte Carlo simulations with $\hat{X}_j$ in lieu of X_j; the estimate of p is

$$p_{est} = \frac{1}{R}\sum_{r=1}^{R} w\left(\hat{X}_0^r, ..., \hat{X}_d^r\right) 1_{\{\hat{X}_0^r + ... + \hat{X}_d^r > a\}} \qquad (7.20)$$

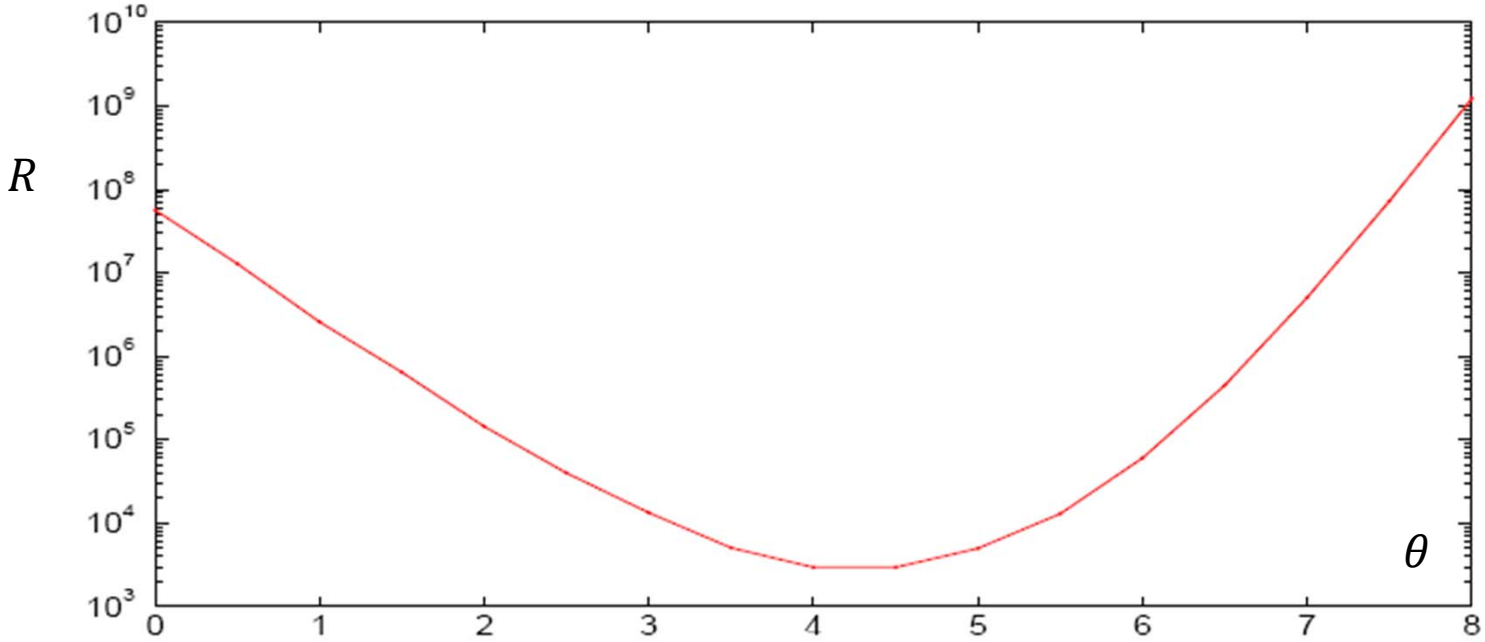
- We do this for several values of θ and find the same estimate $p \approx 6.45\ 10^{-6}$
- What is different ?

Importance Sampling Monte Carlo

We perform R Monte Carlo simulations with $\hat{X}_j$ in lieu of X_j; the estimate of p is

$$p_{est} = \frac{1}{R}\sum_{r=1}^{R} w\left(\hat{X}_0^r, ..., \hat{X}_d^r\right) 1_{\{\hat{X}_0^r + ... + \hat{X}_d^r > a\}} \qquad (7.20)$$

- We do this for several values of θ and find the same estimate $p \approx 6.45\ 10^{-6}$
- What is different ? Hopefully R, the number of runs



Choosing an Importance Sampling Distribution

- What is a good importance sampling distribution ?
  One that minimizes the number of runs

- This can be quantified with the variance of the importance sampling estimator

More formally, we can evaluate the efficiency of an importance sampling estimator of p by its variance

$$\hat{v} = \text{var}\left(\phi(\hat{X})w(\hat{X})\right) = \mathbb{E}\left(\phi(\hat{X})^2 w(\hat{X})^2\right) - p^2$$

Assume that we want a $1-\alpha$ confidence interval of relative accuracy β. By a similar reasoning as in Eq.(7.14), the required number of Monte Carlo estimates is

$$R = \hat{v}\frac{\eta^2}{\beta^2 p^2} \tag{7.21}$$

Thus, it is proportional to $\hat{v}$. In the formula, η is defined by $N_{0,1}(\eta) = 1 - \frac{\alpha}{2}$; for example, with $\alpha = 0.05, \beta = 0.1$, we need $R \approx 400\hat{v}/p^2$.

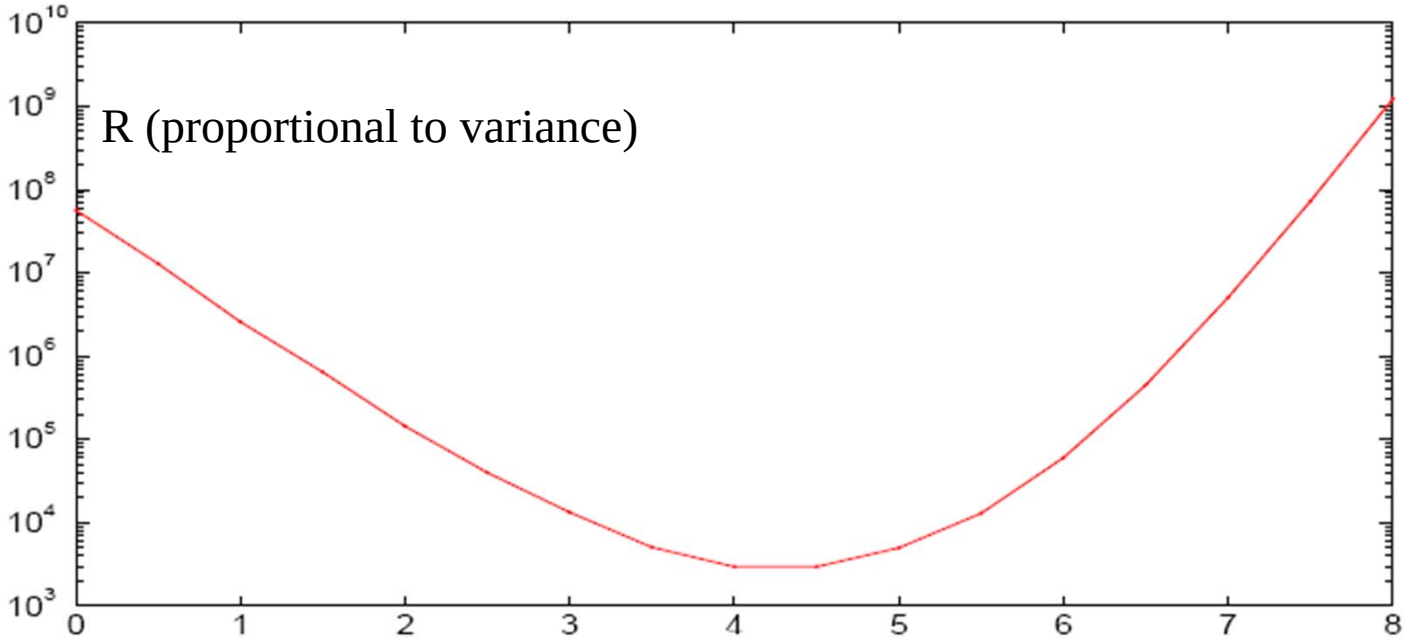
EXAMPLE 7.17: BIT ERROR RATE, RE-VISITED. We can apply Algorithm 2 directly. With the same notation as in Example 7.16, an estimate of $\hat{v}$, the variance of the importance sampling estimator, is

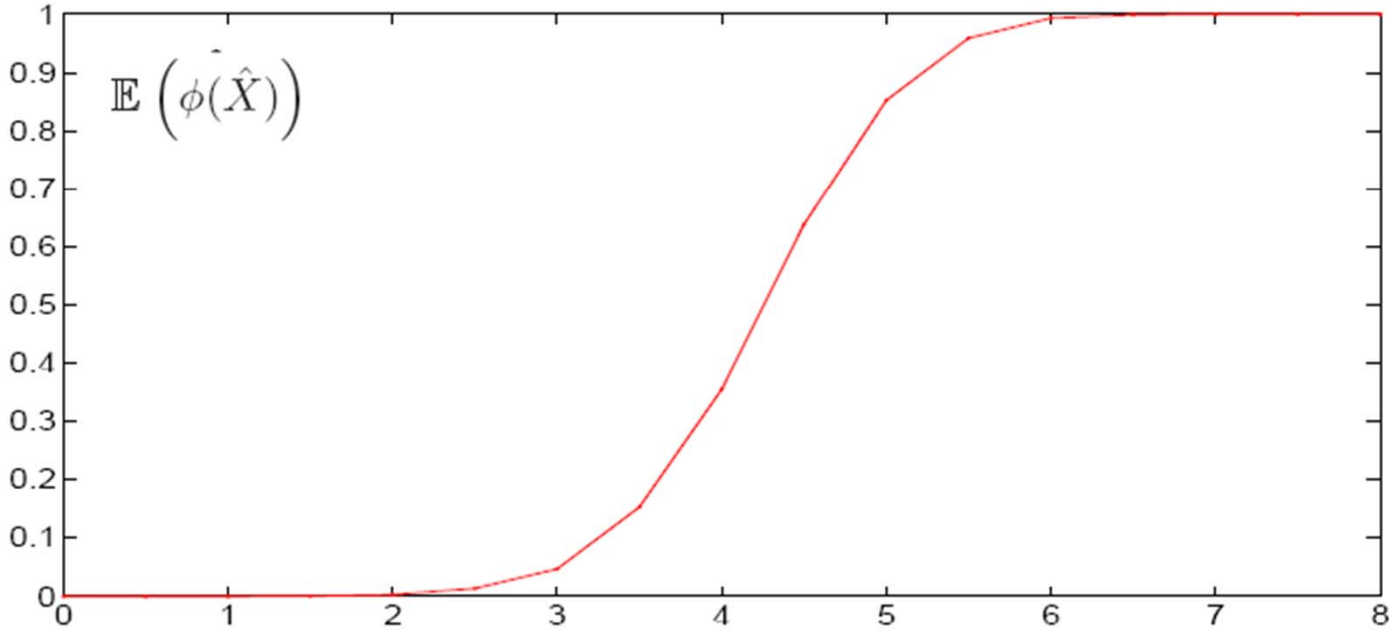
$$\hat{v}_{est} = \frac{1}{R}\sum_{r=1}^{R} w\left(\hat{X}_0^r, ..., \hat{X}_d^r\right)^2 1_{\{\hat{X}_0^r + ... + \hat{X}_d^r > a\}} - p_{est}^2 \tag{7.22}$$

We computed $\hat{v}_{est}$ for different values of θ; Figure 7.12 shows the corresponding values of the required number of simulation runs R (to reach 10% accuracy with confidence 95%), as given by Eq.(7.21)).

- The smallest variance is for

$$\mathbb{E}\left(\phi(\hat{X})\right) \approx 0.5$$





Choosing an Importance Sampling Distribution (1)

- Rule of thumb:
  - The events of interest, under the importance sampling distribution should be

    not rare

    not certain

Choosing an Importance Sampling Distribution (2)

- The optimal importance sampling distribution is the one that minimizes

$$\mathbb{E}\left(\phi(\hat{X})^2 w(\hat{X})^2\right)$$

- Is this the same as minimizing the variance of the importance sampling estimator ?

function MAIN

  $\eta = 1.96$; $\beta = 0.1$; pCountMin= 10; ▷ β is the relative accuracy of the final result

  GLOBAL $R_0 = 2\frac{\eta^2}{\beta^2}$; ▷ Typical number of iterations

  ▷ R_0 chosen by Eq.(7.14) with $p = 0.5$

  $R_{\max} = 1E + 9$; ▷ Maximum number of iterations

  $c = \frac{\beta^2}{\eta^2}$;

  Find $\theta_0 \in \Theta$ which minimizes varest(θ);

  pCount0= 0; pCount= 0; $m_2 = 0$;

  for $r = 1 : R_{\max}$ **do**

    draw a sample x of $\hat{X}$ using parameter θ_0;

    pCount0=pCount0+$\phi(x)$;

    pCount=pCount+$\phi(x)w(x)$;

    $m_2 = m_2 + (\phi(x)w(x))^2$;

    if $r \geq R_0$ and pCountMin $<$ pCount $<$ $r-$ pCountMin **then**

      $p = \frac{\text{pCount}}{r}$;

      $v = \frac{m_2}{r} - p^2$;

      if $v \leq cp^2r$ **then break**

      end if

    end if

  end for

  return p, r

end function

function VAREST(θ) ▷ Test if $\mathbb{E}\left(\phi(\hat{X})\right) \approx 0.5$ and if so estimate $\mathbb{E}\left(\phi(\hat{X})^2 w(\hat{X})^2\right)$

CONST $\hat{p}_{\min} = 0.3$, $\hat{p}_{\max} = 0.7$;

GLOBAL R_0;

$\hat{p} = 0$; $m_2 = 0$;

for $r = 1 : R_0$ **do**

draw a sample x of $\hat{X}$ using parameter θ;

$\hat{p} = \hat{p} + \phi(x)$;

$m_2 = m_2 + (\phi(x)w(x))^2$;

end for

$\hat{p} = \frac{\hat{p}}{R}$;

$m_2 = \frac{m_2}{R}$;

if $\hat{p}_{\min} \leq \hat{p} \leq \hat{p}_{\max}$ **then**

return m_2;

else

return ∞;

end if

end function

A Generic Algorithm

- Ideas : empirically find importance sampling distribution such that
  - Average occurrence of event of interest is close to 0.5
  - Minimizes $\mathbb{E}\left(\phi(\hat{X})^2 w(\hat{X})^2\right)$
  - Can be computed by Monte Carlo with small number of runs

- The algorithm does not say how to do one important thing: which one ?

Conclusion

- If you have to simulate rare events, importance sampling is probably applicable to your case and will provide siginificant speedup
- A generic algorithm can be used to find a good sampling distribution