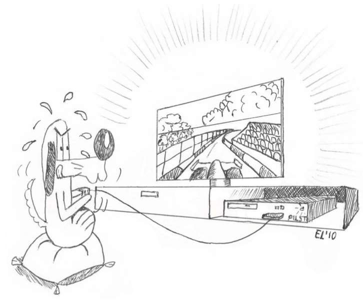
PERFORMANCE EVALUATION
EXERCISES

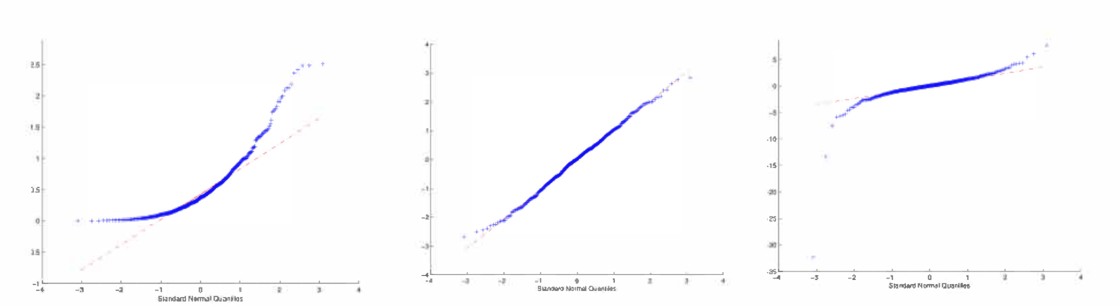
SIMULATION

Jean-Yves Le Boudec, February 18, 2021



In the text below, rand$(1, n)$ produces a sequence of n independent samples of the uniform distribution between 0 and 1 and randn$(1, n)$ produces n independent samples of the standard normal distribution.

1. Which of the following three normal qq-plots is for an exponential distribution ?



   (a) ☐ The first
   (b) ☐ The second
   (c) ☐ The third
   (d) ☐ The first and second
   (e) ☐ The first and third
   (f) ☐ The second and third
   (g) ☐ All
   (h) ☐ None

2. Independent simulation outputs are obtained by...

   A executing the runs on parallel threads using the same seed.

   B executing the runs on parallel threads using truly random seeds.

   C using the last RNG state of one run as seed to the next run.

   (a) ☐ A
   (b) ☐ B
   (c) ☐ C
   (d) ☐ A or B
   (e) ☐ A or C

(f) ☐ B or C

(g) ☐ All

(h) ☐ None

3. The random variable X is integer and its distribution is given by:

$$\begin{aligned} \mathbb{P}(X=-1) &= 0.1 \\ \mathbb{P}(X=0) &= 0.8 \\ \mathbb{P}(X=1) &= 0.1 \end{aligned}$$

Write the pseudo-code of a simulation program to generate a sample of X.

4. The standard Pareto distribution with index 1 has PDF $f_X(x) = \frac{1}{x^2}\mathbf{1}_{\{x>1\}}$. Write the pseudo-code of a simulation program to generate a sample of X.

5. We draw a (X, Y) point uniformly at random in the unit disk. We observe its radius R and its angle Θ.

   (a) Compute the PDFs $f_{R,\Theta}$, f_F and f_Θ.

   (b) Are R and Θ uniform ? Independent ?

6. The function $H()$ returns a positive random number with PDF $f()$ and CDF $F()$; c is a positive constant such that $F(c) < 1$. The following program computes a random variable Y.

```
do forever
    X ← H()
    if X > c
        Y ← X
        return (Y)
```

Does the program always terminate ? Under these conditions, what is the distribution of Y? What are its PDF and CDF ? How many iterations does the program take in average?

7. The time T between successive sending of messages by a sensor is random but always larger than some constant τ. Its average is $\bar{t}$ (with $\bar{t} > \tau$).

   (a) Bart models T as follows:

   $$T = \tau + X, \quad X \sim \text{Exp}(\lambda_1)^1$$

   What are the PDF and CDF of T ? What value should λ_1 have ? Write the pseudo-code of a simulation program to generate a sample of T according to Bart. How many calls to the random number generator does this program use ? Run your program to generate a histogram of 1000 samples; which shape do you expect ?

[1] Exp(λ) stands for the exponential distribution with rate λ.

(b) Lisa models T as a sample of the conditional distribution of Y given $Y > \tau$ with $Y \sim \text{Exp}(\lambda_2)$. What are the PDF and CDF of T ? What value should λ_2 have ? Write the pseudo-code of a simulation program to generate a sample of T using rejection sampling according to Lisa's model. How many calls to the random number generator does this program use ?

(c) Compare these modelling and simulation methods.

8. The following simulation program computes a random vector (X_1, X_2):

$X_1 \leftarrow$ randn$(1, 1)$
$B \leftarrow$ rand$(1, 1)$
    if $B > 0.5$
        $X_2 \leftarrow X_1$
    else
        $X_2 \leftarrow -X_1$
return(X_1, X_2)

(a) What is the distribution of X_1 ? of X_2 ?

(b) What is the distribution of $X_1 + X_2$? What are its PDF and CDF ?

(c) Plot 1000 samples of (X_1, X_2), what do you see ? Compare to a plot of 1000 samples of (Y_1, Y_2) where Y_1 and Y_2 are independent and standard normal. Is (X_1, X_2) a gaussian random vector ?

9. The random variable X is nonnegative and has complementary CDF $F^c(x) = \mathbb{P}(X > c) = (1 + x)e^{-x}1_{x>0}$. What is the PDF of X ? Write the pseudo-code of a simulation program to generate a sample of X.

10. We want to simulate a random point (X, Y) in the unit disk, whose PDF is

$$f_{X,Y}(x, y) = \eta\sqrt{x^4 + y^4}\mathbf{1}_{\{x^2+y^2\leq 1\}}$$

In the formula,η is a normalizing constant. Give the pseudo-code of a program that produces a sample of (X, Y). Try it and sample 10'000 points. How does this visually compare to the uniform distribution on the unit the disk ?

11. What does this program compute ? (A is a subset of $[0; 1]^n$).

$N \leftarrow 0$
do $r = 1 : R$
    if rand$(n, 1) \in A$
        $N \leftarrow N + 1$
return (N/R)