PERFORMANCE EVALUATION EXERCISES

CONFIDENCE INTERVALS

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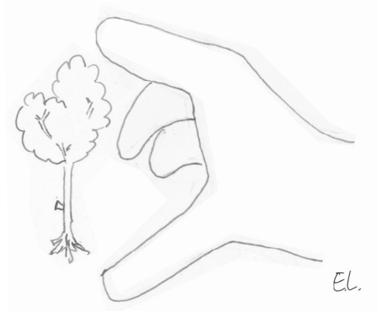

1. **Probability Drill**

The following table gives the complementary CDF of the standard normal distribution.

z	0	1	2	3	4	5
$\mathbb{P}(Z > z)$	0.5	0.1587	0.02275	0.001350	3.167E-05	2.867E-07
z	6	7	8	9	10	11
$\mathbb{P}(Z > z)$	9.866E-10	1.280E-12	6.2216E-16	1.129E-19	7.620E-24	1.911E-28

(a) X is a gaussian random variable with mean μ and standard deviation σ. X measures a time and is in seconds. In which units are μ and σ ?

Which of the following random variables is standard gaussian (= standard normal) ?

i. ☐ $Z = \sigma X + \mu$
ii. ☐ $Z = \frac{X-\mu}{\sigma}$
iii. ☐ $Z = \frac{X}{\sigma} - \mu$
iv. ☐ None of these.

(b) $X_1, ...X_n$ are independent random variables with same expectation μ and same standard deviation σ. What are the expectation and the standard deviation of $\bar{X} = \frac{1}{n}(X_1 + ... + X_n)$?

(c) Lisa models the result of an experiment that produces only positive numbers as a gaussian random variable X with mean $\mu = 10$ and standard deviation $\sigma = 1$. Bart observes that a gaussian distribution may take negative values and claims that Lisa should not use this model. Who is right ?

(d) The pdf of the gaussian distribution mean μ and standard deviation σ is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Give the values of

i. $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

ii. $\int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

iii. $\int_{-\infty}^{+\infty} \frac{x^2}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

(e) X_1 is a gaussian random variable with mean μ_1 and variance σ_1^2; X_2 is a gaussian random variable with mean μ_2 and variance σ_2^2; X_1 and X_2 are independent. What is the pdf of X_1+X_2 ?

(f) $X_1, X_2, ..., X_n$ are independent random variables with values in $\{0, 1\}$; X_i represents a success at the ith experiment, and $p = \mathbb{P}(X_i = 1)$ for all i. $N = X_1 + ... + X_n$ represents the total number of successes out of n experiments. What is the distribution of N? what is its mean and standard deviation ? For large n, can you approximate it with a continuous distribution ?

2. Here are the results of 10 independent simulation runs (throughput in Mb/s):

   13.3 12.9 13.1 13.5 12.8 12.9 13.2 189.6 12.9 13.0

   Give a 95% confidence interval.

   (a) ☐ [12.9 ; 189.6]
   (b) ☐ [12.9 ; 13.5]
   (c) ☐ [13.0 ; 13.1]
   (d) ☐ [0 ; 200]
   (e) ☐ [12.9 ; 63.55]
   (f) ☐ [-2.11 ; 63.55]

3. We have tested a system for errors and found 0 error in 36 runs. Give a confidence interval for the probability of failure.

   (a) ☐ [0% ; 0%]
   (b) ☐ [0%; 1.74%]
   (c) ☐ [0% ; 9.74%]
   (d) ☐ [0% ; 33.74%]
   (e) ☐ [0 ; 100%]

4. We expect ...

   (a) ☐ ... a 95% confidence interval to be narrower than a 99% confidence interval
   (b) ☐ ... a 95% confidence interval to be wider than a 99% confidence interval
   (c) ☐ It depends on the data
   (d) ☐ It depends on the type of confidence interval

5. A data set $x_i > 0$ is such that $y_i = 1/x_i$ looks normal. A 95% confidence interval for the mean of y_i is $[L; U]$. Is it true that a confidence interval for the mean of x_i is $[1/U; 1/L]$?

   (a) ☐ Yes
   (b) ☐ No

6. A data set $x_i > 0$ is such that $y_i = 1/x_i$ looks normal. A 95% confidence interval for the median of y_i is $[L'; U']$. Is it true that a confidence interval for the median of x_i is $[1/U'; 1/L']$?

   (a) ☐ Yes
   (b) ☐ No

7. We have obtained $n = 100$ independent measurements of response time. The values, in msec, sorted in increasing order, are:

18.0062 18.7358 18.7527 19.3283 19.3893 19.6023 20.0612 20.1427 20.2037 20.4588
... (8 lines not shown) ...
25.0155 25.0720 25.2769 25.2907 25.9579 26.1526 26.7172 26.7557 27.3556 27.8489

The mean of the measurements is $m = 22.8705$ and the standard deviation is $s = 1.8840$. Give a prediction interval with confidence level 95%.

8. The paper below is available on moodle (and elsewhere).

   Prytz, G. and Johannessen, S., 2005, September. Real-time performance measurements using UDP on Windows and Linux. In Emerging Technologies and Factory Automation, 2005. ETFA 2005. 10th IEEE Conference on (Vol. 2, pp. 8-pp). IEEE.

   Table 1 of this paper gives measurements with uncertainty bounds. For example, the first two rows give:

Time (μs)	Processor Load	Network Speed	
		100 Mbps	1 Gbps
UDP output stack time	Low	$15 \pm 2.3\mu$s	$16 \pm 1.9\mu$s
	High	$91 \pm 29\mu$s	$80 \pm 36\mu$s

   Are these intervals correctly computed ? If so, what do they represent ?

   Compute confidence intervals at level 95% for the first two rows of table 1. Gives results in the same format as the table above.

9. We measure the round trip propagation time over a fiber infrastructure as follows. First we measure the delay Z over one fiber, then we measure the delay T over a second fiber; our result is $R = Z + T$.

To obtain a good estimate of Z we perform m independent measurements and obtain $z_1, ..., z_m$; we also perform n independent measurements of T and obtain $t_1, ..., t_n$. Both m and n are very large and $m \neq n$.

Assume that Z and T are random variables with finite mean and variance. Find a formula for a confidence interval for $\mu_R = \mu_Z + \mu_T$.

(Hint: use inspiration from the slide "Idea of Proof" that follows the description of confidence interval for the mean, asymptotic case.)