ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Network Calculus

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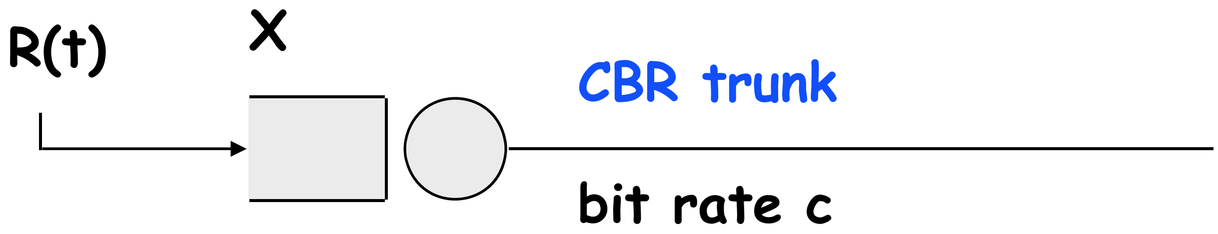
Lecture Plan

- Introduction

  Arrival curves, Service curves, Pay Bursts only Once, IntServ and ATM
- Shapers, shapers keep arrival constraints
- Packetizers. Fixed Point Theorem and Residuation.

  GR nodes and schedulers.
- Aggregate Multiplexing and Instability. Explicit bounds. Diff-Serv
- Optimal Smoothing. Video Playback

A simple example



- assume
  - $R(t)$ = sum of arrived traffic in $[0, t]$ is known
- (Reich:) required **buffer** for a bit rate c is

$$\sup_{s \leq t} \{R(t) - R(s) - c\,(t-s)\}$$

What is Network Calculus ?

- Deterministic analysis of queuing / flow systems arising in communication networks
- Abstraction of schedulers
  - systems that are not a single queue, Reich's formula does not apply
- Uses min, max for binary operators and integrals (min-plus / max-plus algebra)

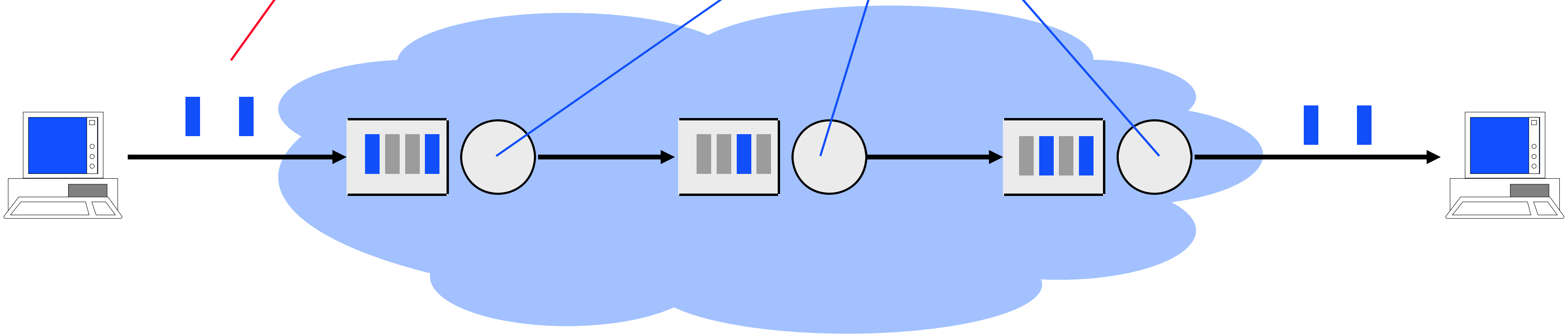
Examples

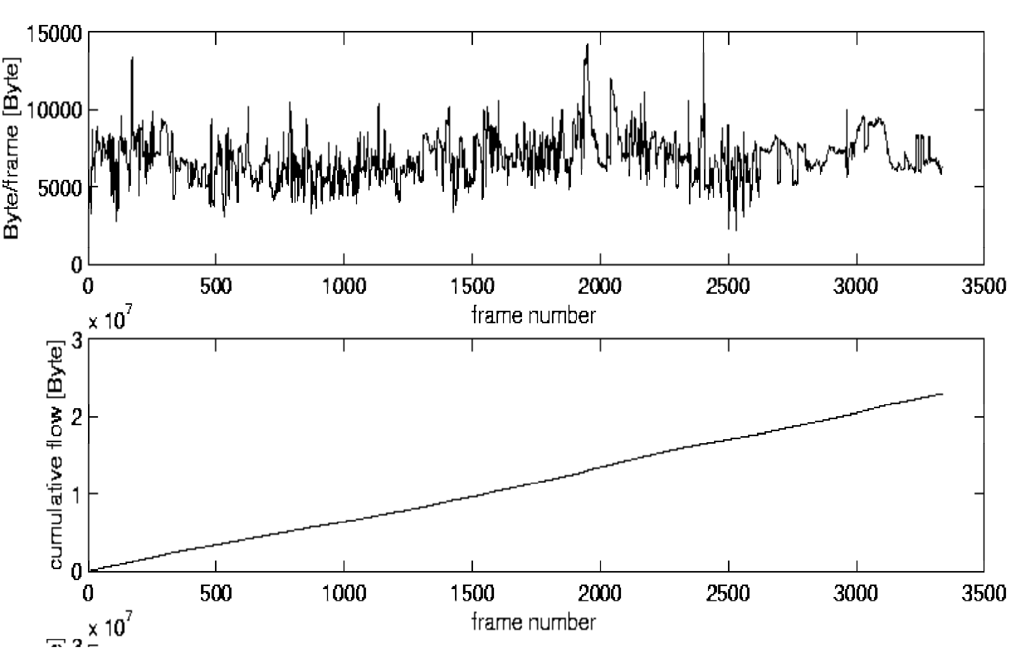
- Pay Bursts only Once
- Re-Shaping is For Free
- Minimum Playout Buffer
- Diff-Serv delay dimensioning

Day 1

Arrival and Service Curves

- Internet integrated services use the concepts of *arrival curve* and *service curves*

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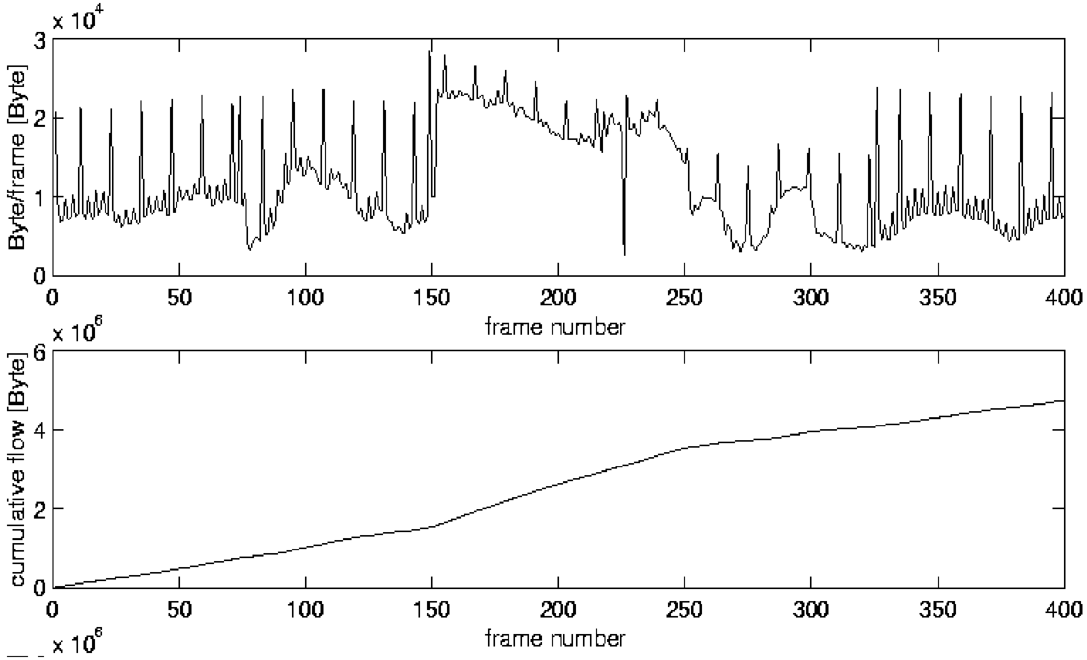
cumulative flow [Byte]
frame number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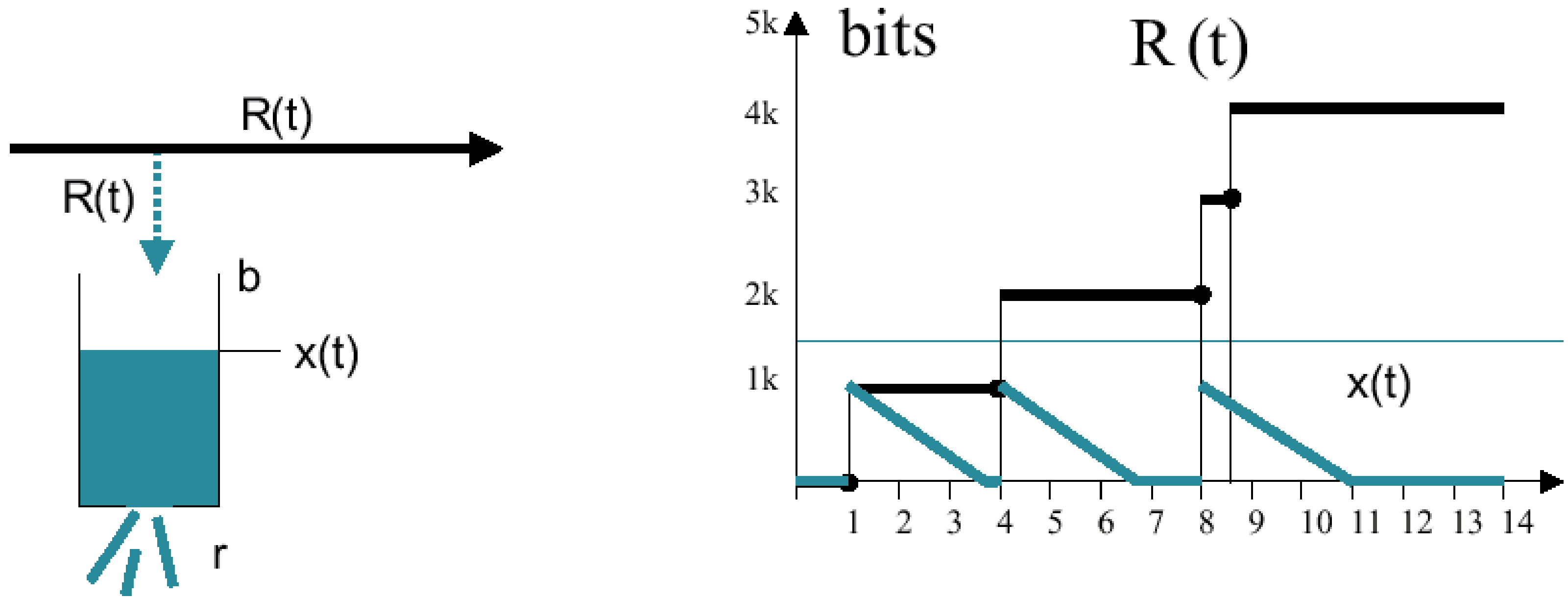
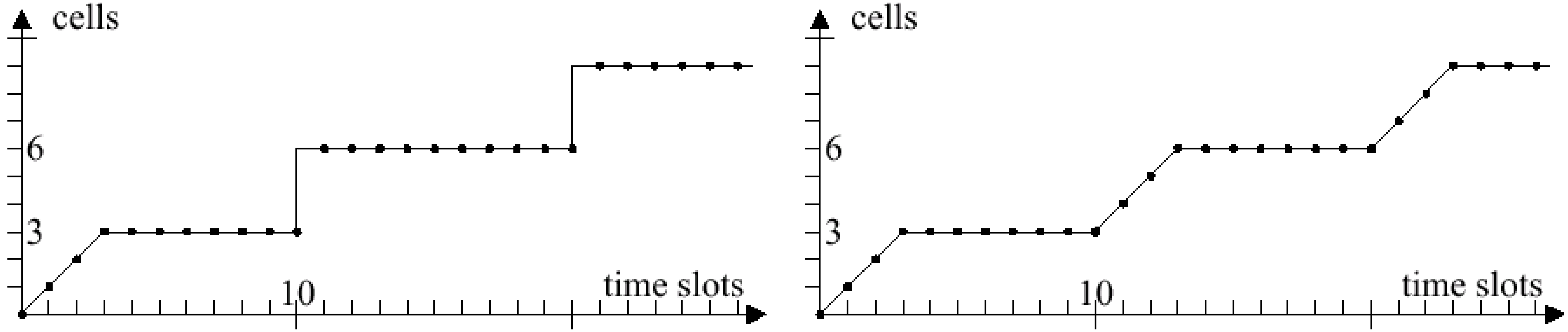


Figure 1.4: A Leaky Bucket Controller. The second part of the figure shows (in grey) the level of the bucket $x(t)$ for a sample input, with $r = 0.4$ kbits per time unit and $b = 1.5$ kbits. The packet arriving at time $t = 8.6$ is not conformant, and no fluid is added to the bucket. If b would be equal to 2 kbits, then all packets would be conformant.

arrival time	0	10	18	28	38	48	57
tat before arrival	0	10	20	30	40	50	60
result	c	c	c	c	c	c	non-c

arrival time	0	10	15	25	35
tat before arrival	0	10	20	20	30
result	c	c	non-c	c	c


cells
6
3
10
time slots
cells
6
3
10
time slots

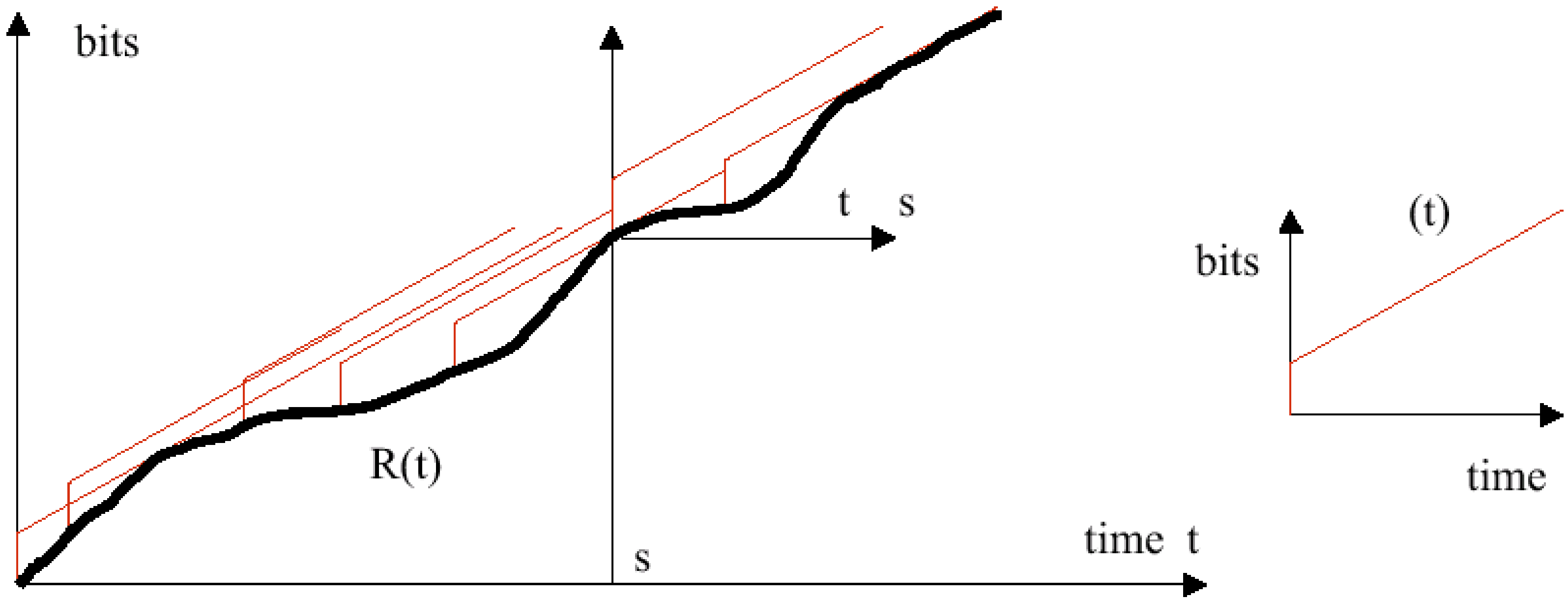
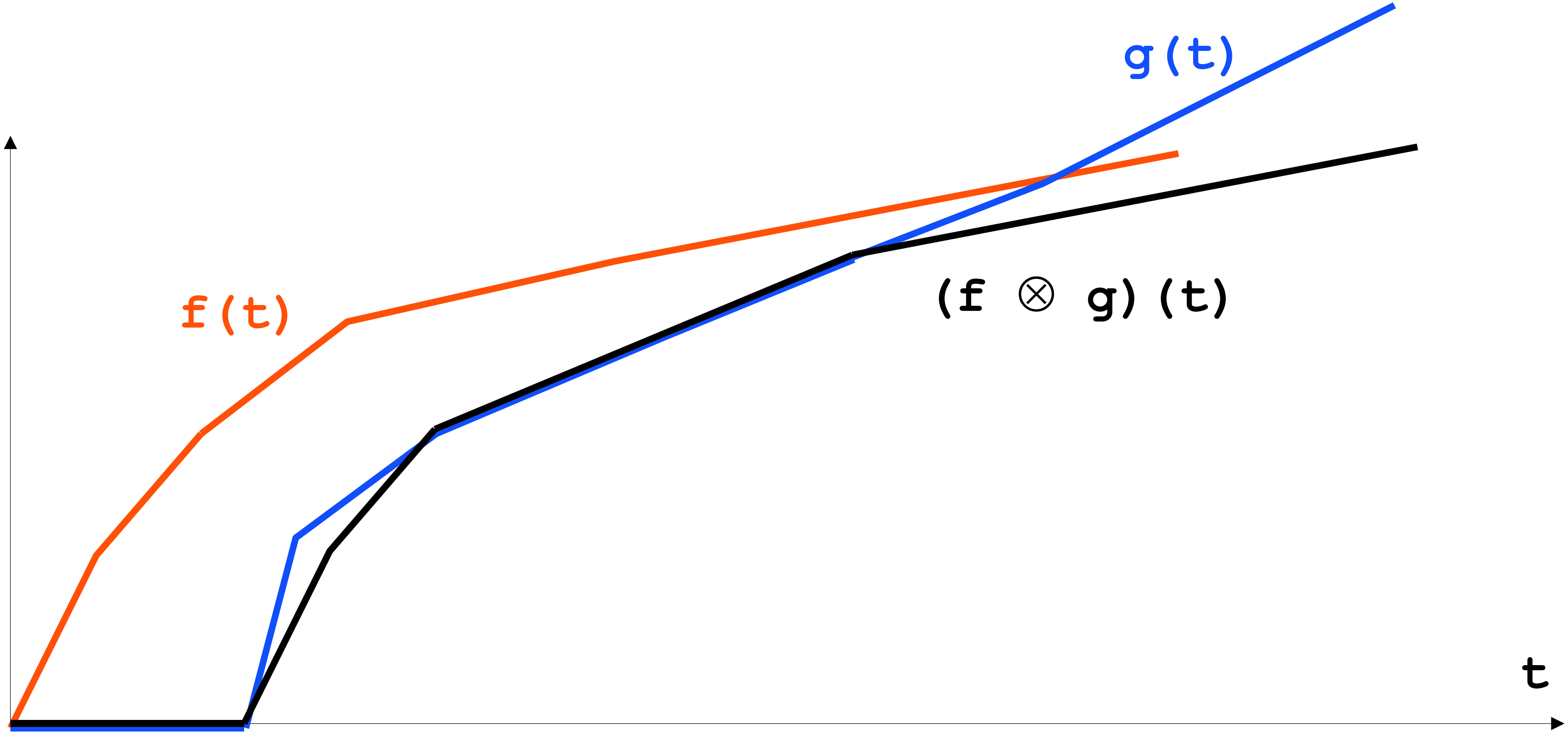


Figure 1.3: Example of Constraint by arrival curve, showing a cumulative function $R(t)$ constrained by the arrival curve $\alpha(t)$.

Min-plus convolution $\otimes$

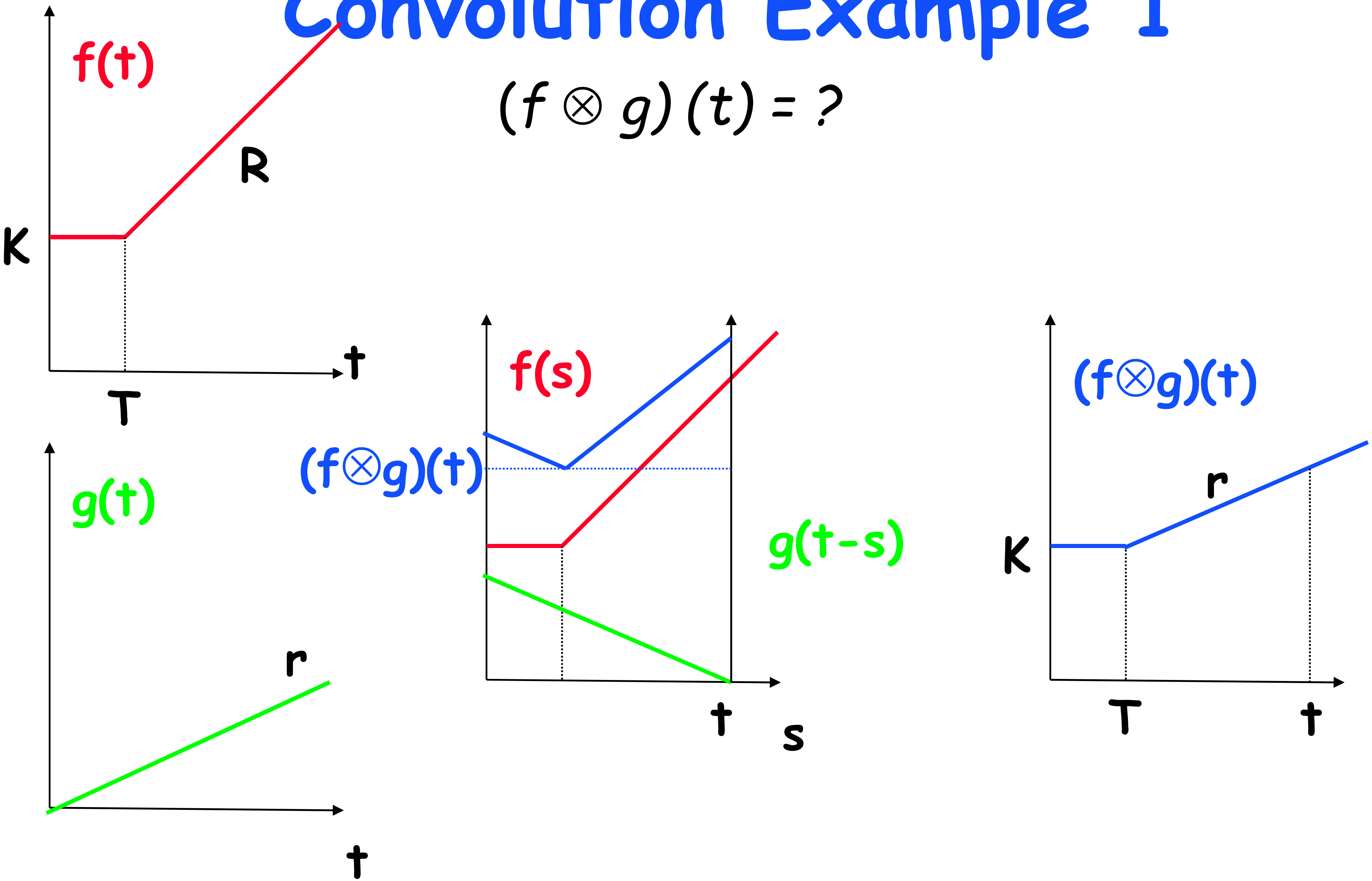
- Definition

$$(f \otimes g)(t) = \inf_u \{ f(t-u) + g(u) \}$$

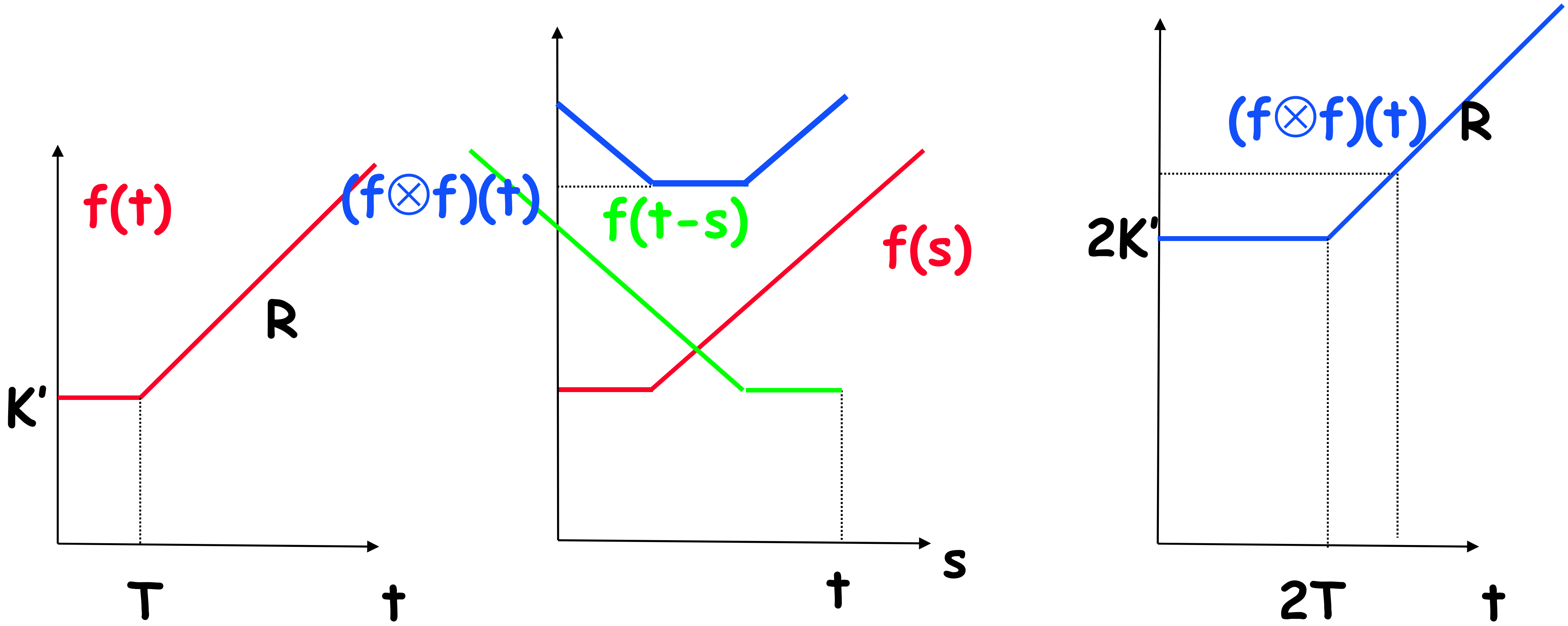


Convolution Example 1

$$(f \otimes g)(t) = ?$$



Convolution Example 2

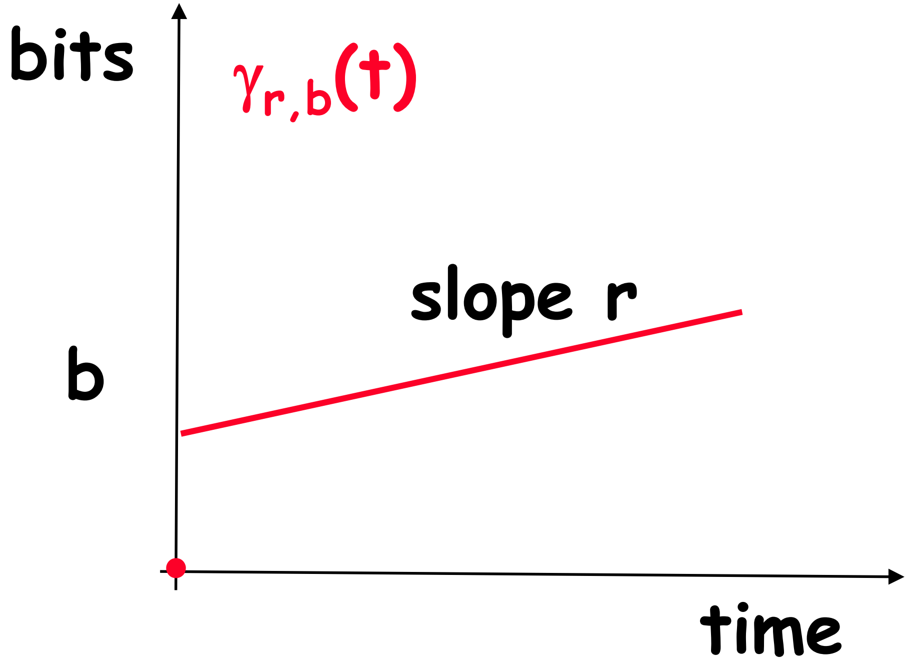


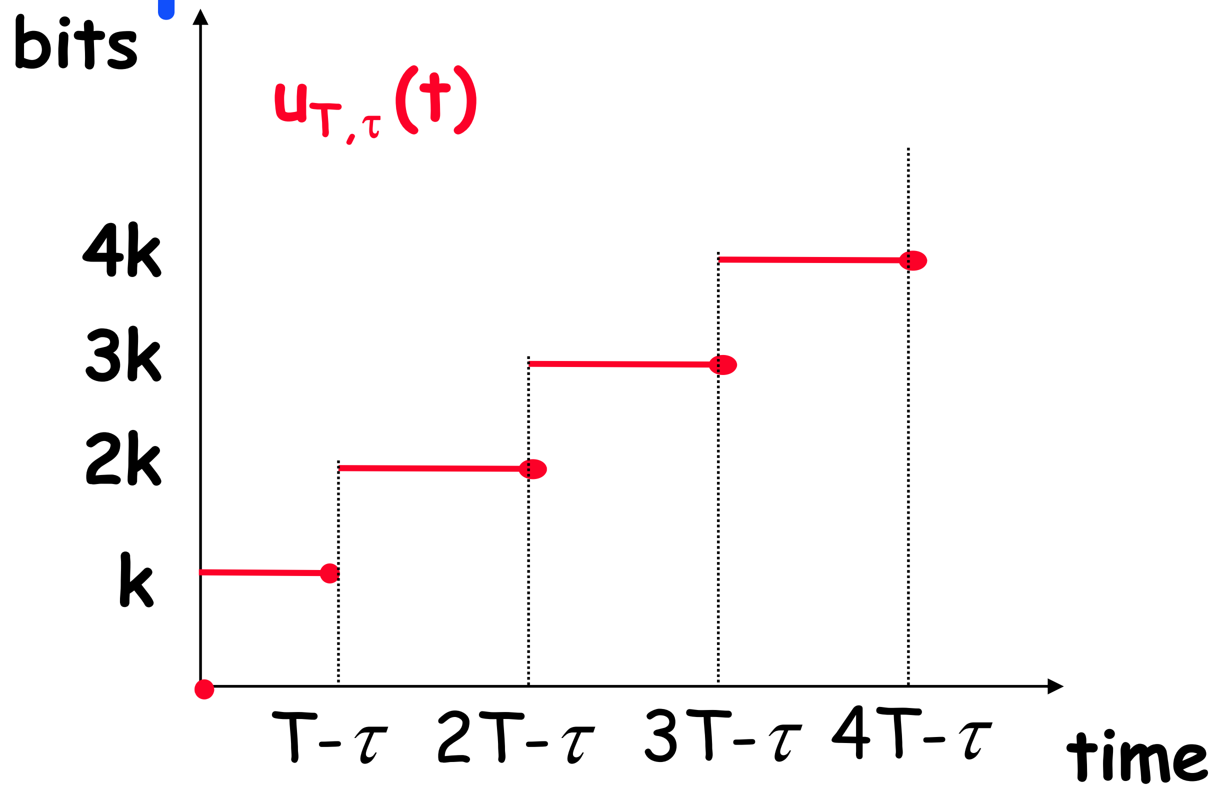
$$(f \otimes f)(t) = ?$$

Star-shaped, concave, convex functions

- f is concave
  - $\Leftrightarrow \forall\ 0 \le u \le 1,\ f(ux + (1-u)y) \ge uf(x) + (1-u)f(y)$
- f is convex
  - $\Leftrightarrow -f$ is concave
- f is star-shaped
  - $\Leftrightarrow f(t)/t \le f(s)/s \quad \forall\ s \le t$
- f is concave $\not\Rightarrow$ f is star-shaped

Examples



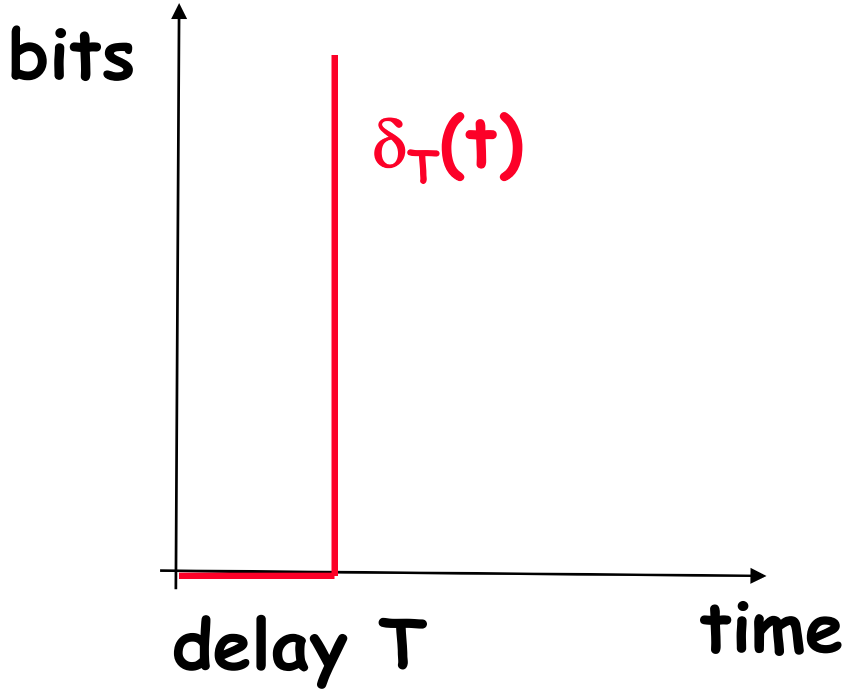


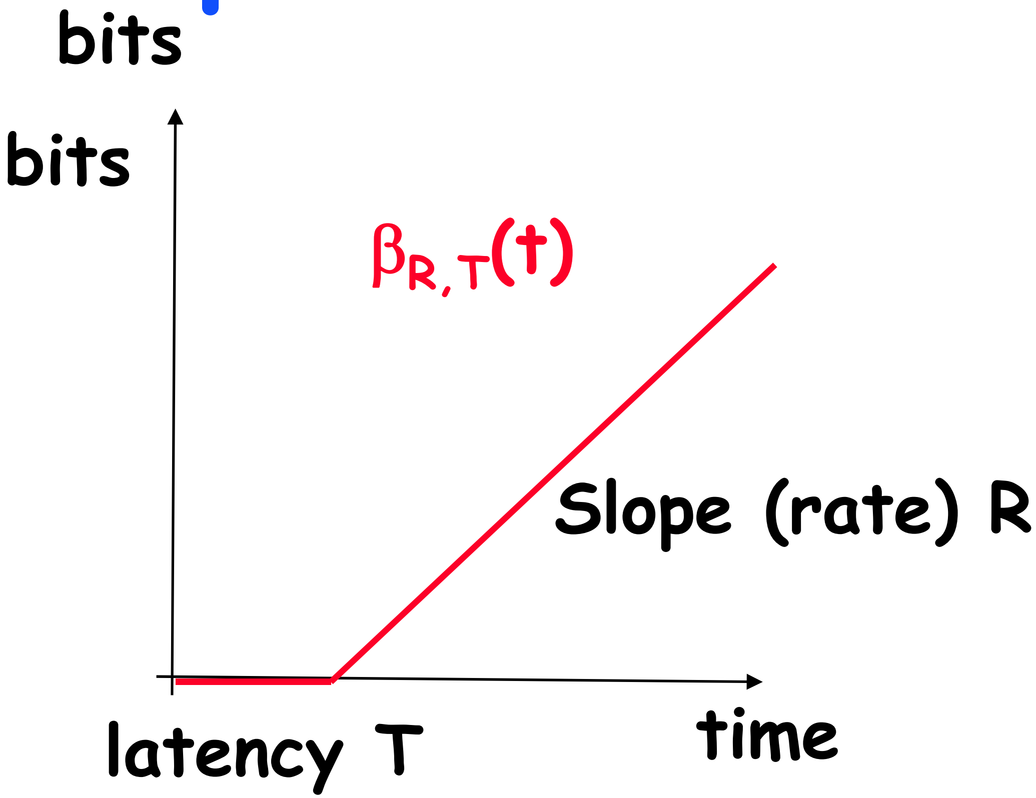
$\gamma_{r,b}$ is concave

is $u_{T,\tau}$ star-shaped ?

concave ?

Examples

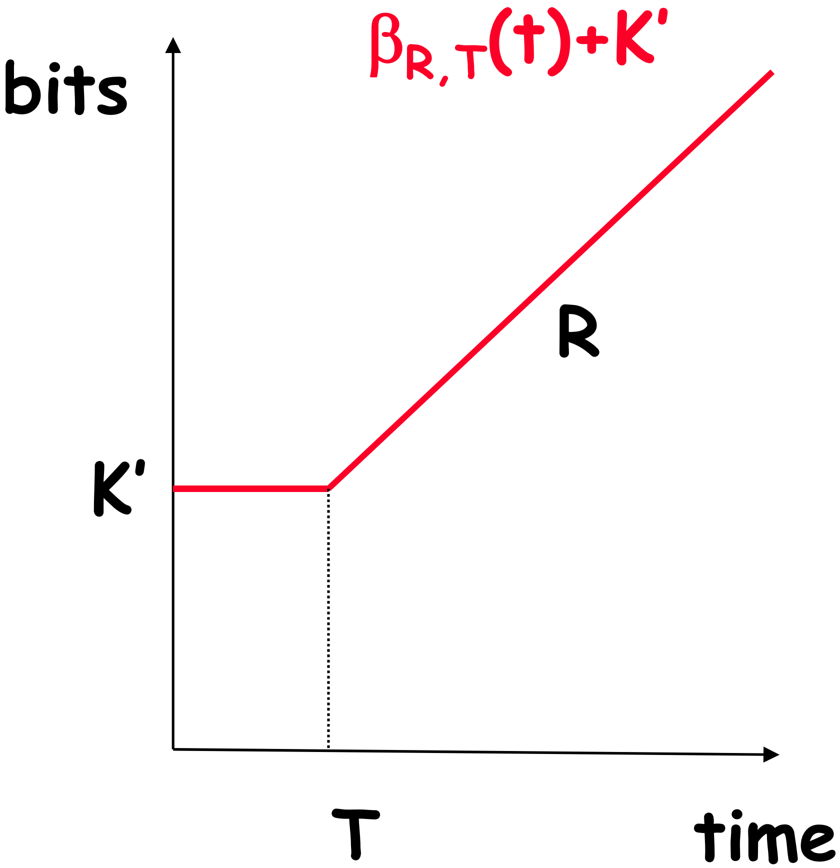


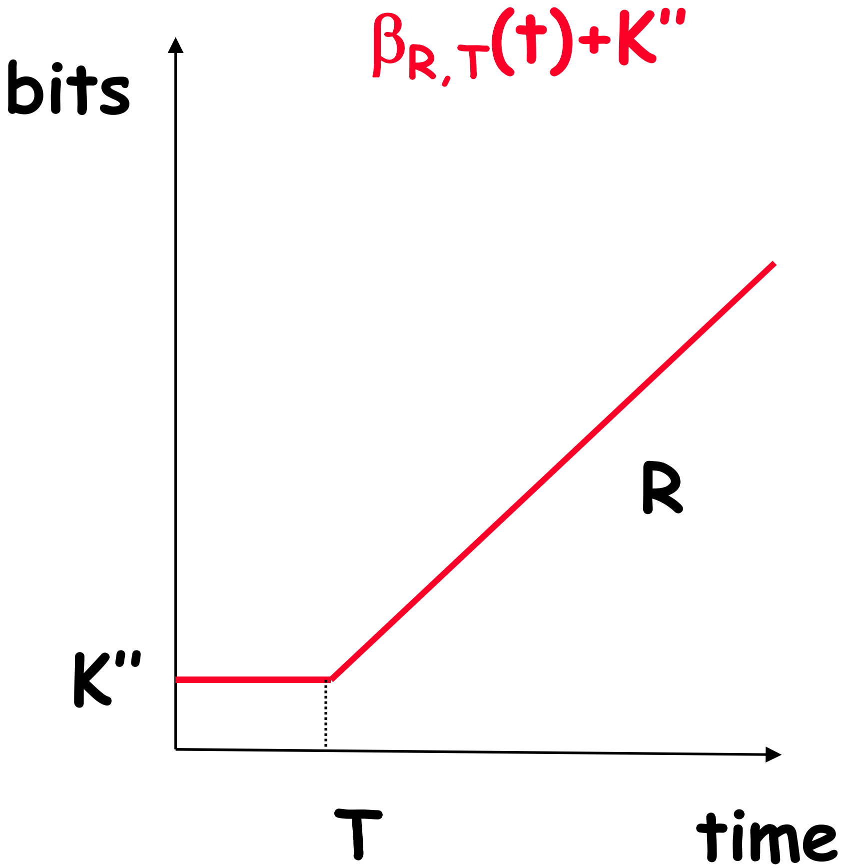


$\beta_{R,T}$ is convex

δ_T is convex

Examples





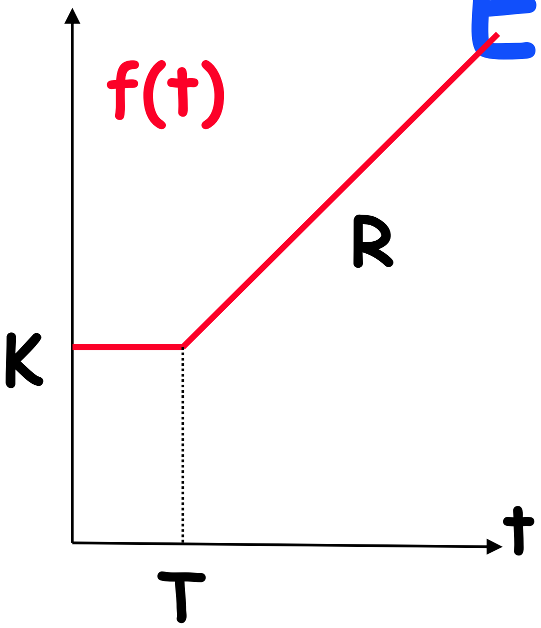
is $\beta_{R,T} + K'$ star-shaped ?
concave ?

is $\beta_{R,T} + K''$ star-shaped ?
concave ?

Some properties of min-plus convolution

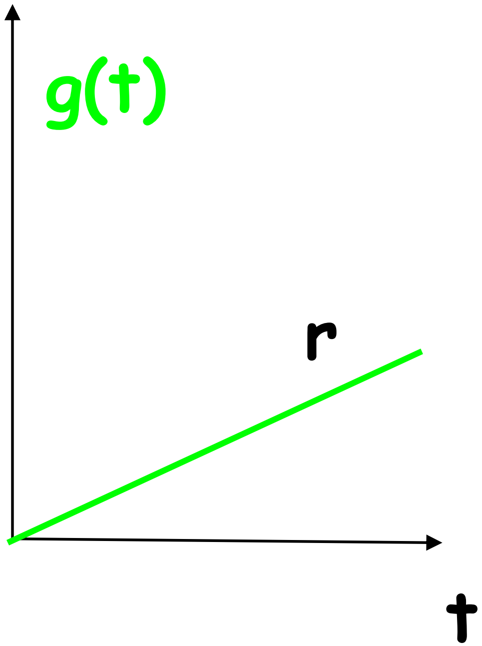
- $\otimes$ is associative
- $\otimes$ is commutative
- $K + (f \otimes g) = (K + f) \otimes g$ when K is a constant
- $f \otimes \delta_T (t) = f(t-T)$
  - $\delta_T(t) = 0$ for $t \leq T$ and $\delta_0(t) = \infty$ for $t > T$
- Star-shaped (concave) functions passing through the origin: $f \otimes g = f \wedge g$

Example 1 (using rules)

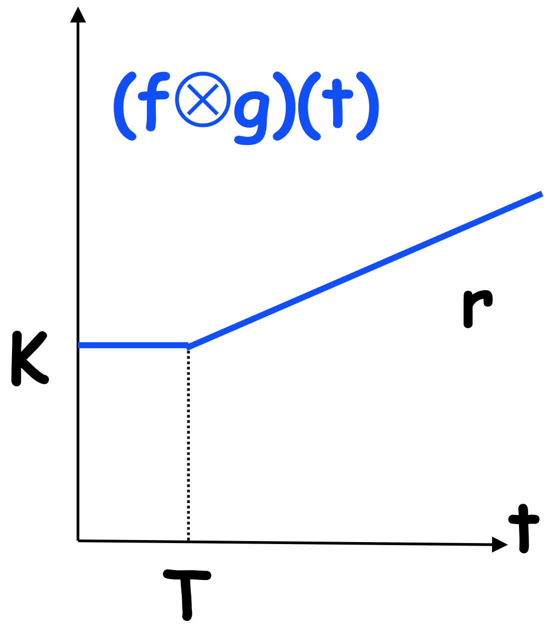


$$f = K + \beta_{R,T}$$
$$= K + \delta_T \otimes \lambda_R$$

$$\begin{aligned} f \otimes g &= (K + \delta_T \otimes \lambda_R) \otimes \lambda_r \\ &= K + ((\delta_T \otimes \lambda_R) \otimes \lambda_r) \\ &= K + (\delta_T \otimes (\lambda_R \otimes \lambda_r)) \\ &= K + (\delta_T \otimes (\lambda_R \wedge \lambda_r)) \\ &= K + (\delta_T \otimes \lambda_r) \\ &= K + \beta_{r,T} \end{aligned}$$

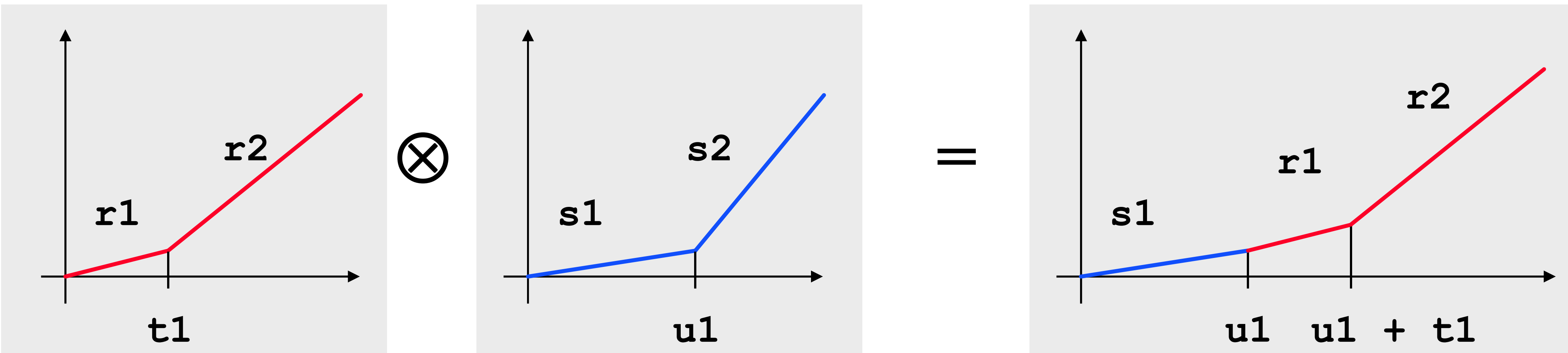


$g = \lambda_r$ concave
with $g(0) = 0$

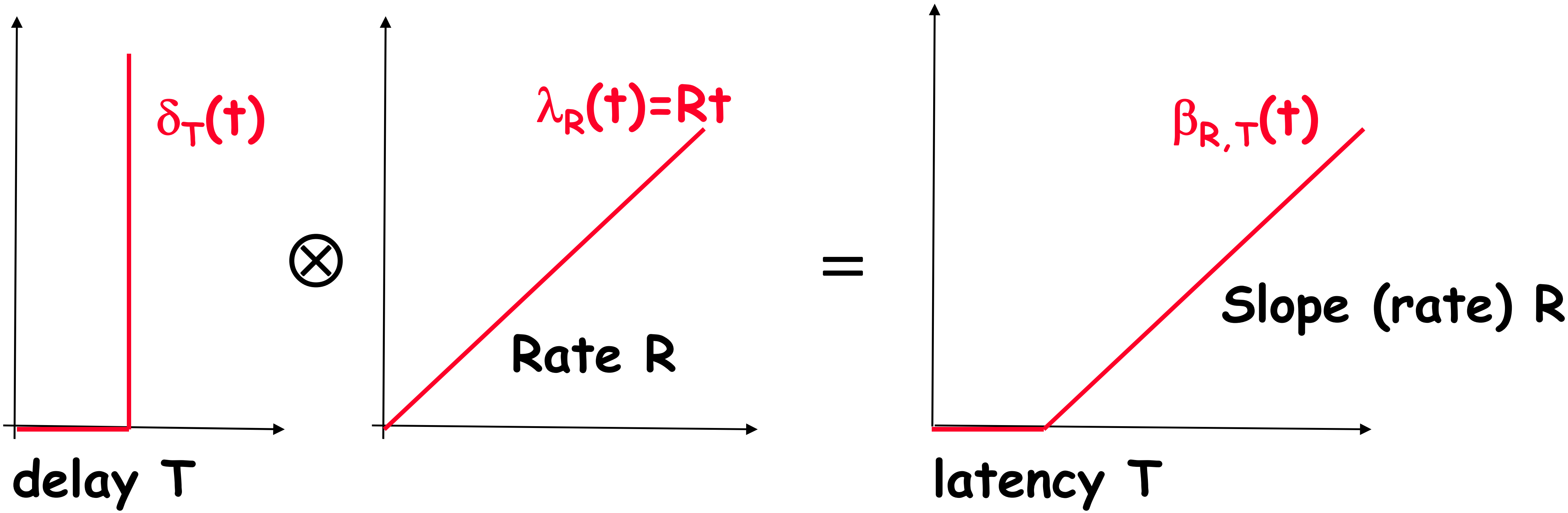


Min-Plus convolution rule for convex functions

- convex piecewise linear wide-sense increasing, passing by origin: put segments end to end with increasing slope



Example: rate latency function



δ_T is convex (delay function)

λ_R is convex (delay function)

$\beta_{R,T}$ is convex (rate-latency function)

- we could also use: $f \otimes \delta_T\,(t) = f\,(t-T)$

We can express arrival curves with min-plus convolution

- Arrival Curve property means for all $0 \leq s \leq t$,

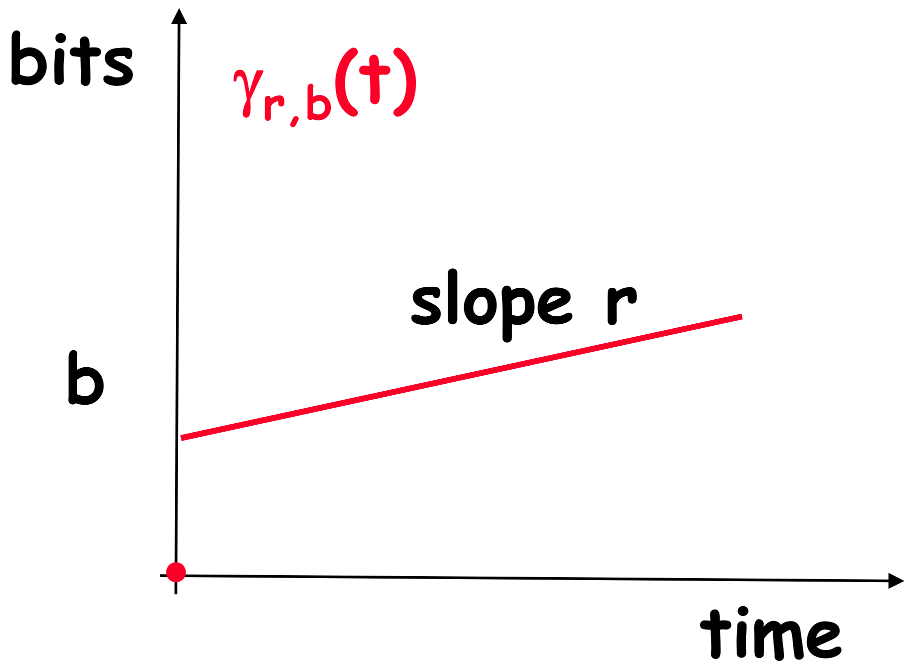
$$x(t) - x(s) \leq \alpha(t-s)$$

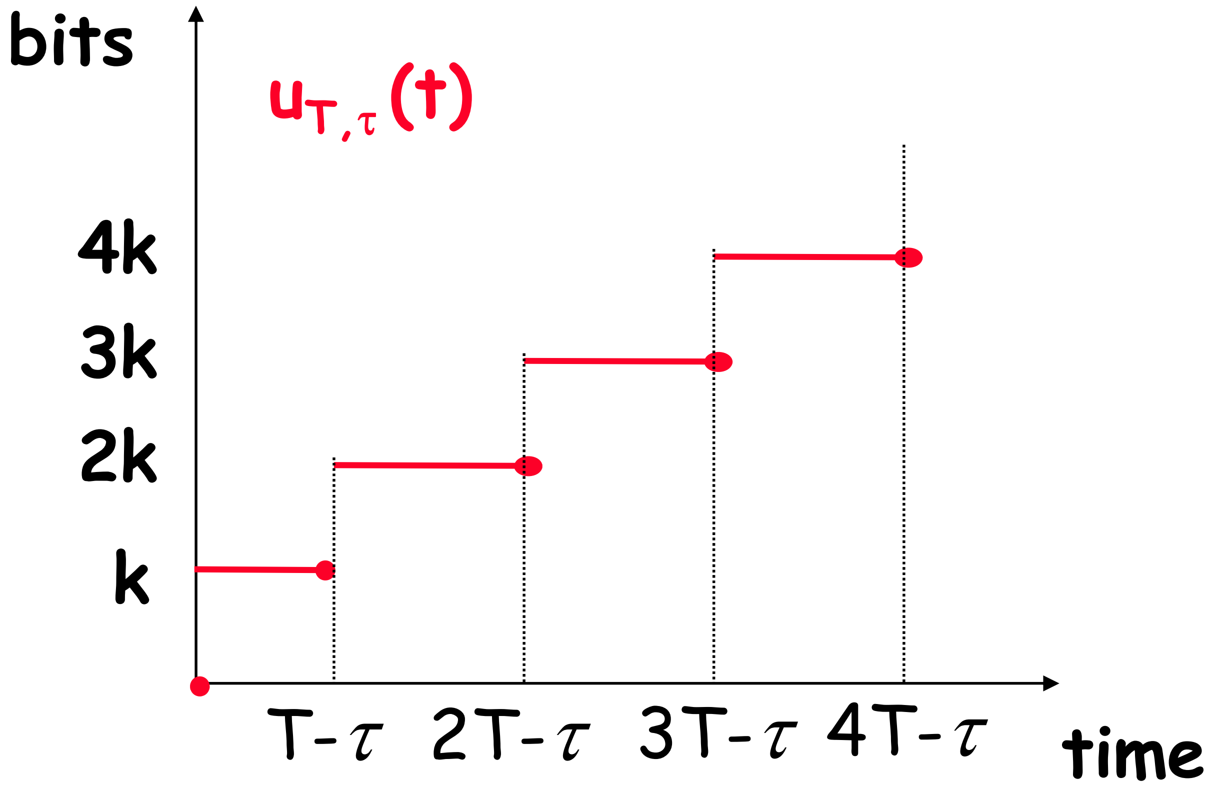
<-> $x(t) \leq x(s) + \alpha(t-s)$ for all $0 \leq s \leq t$

<-> $x(t) \leq \inf_u \{ x(u) + \alpha(t-u) \}$

<-> $x \leq x \otimes \alpha$

Sub-additive functions

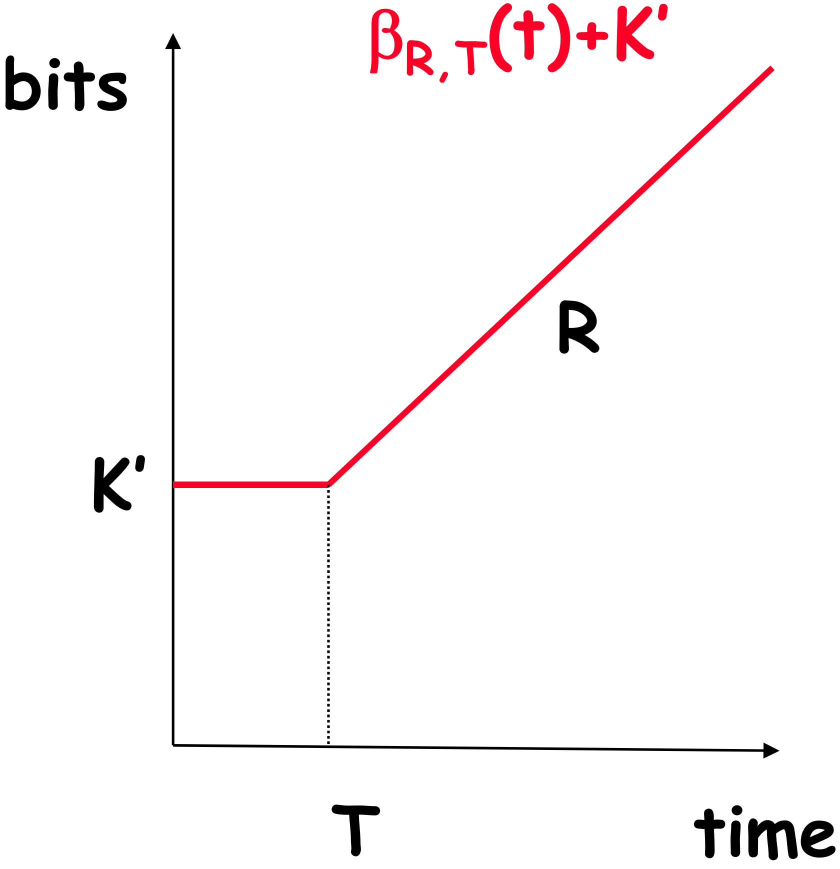


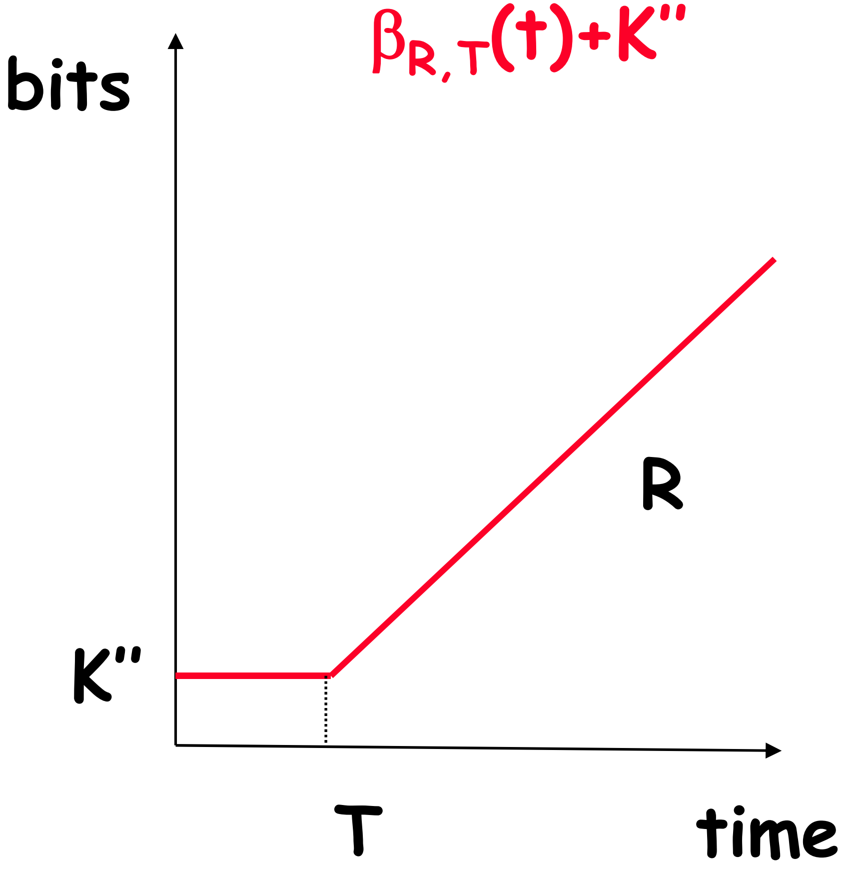


is $\gamma_{r,b}$ sub-additive ?

is $u_{T,\tau}$ sub-additive ?

Sub-additive closures ?

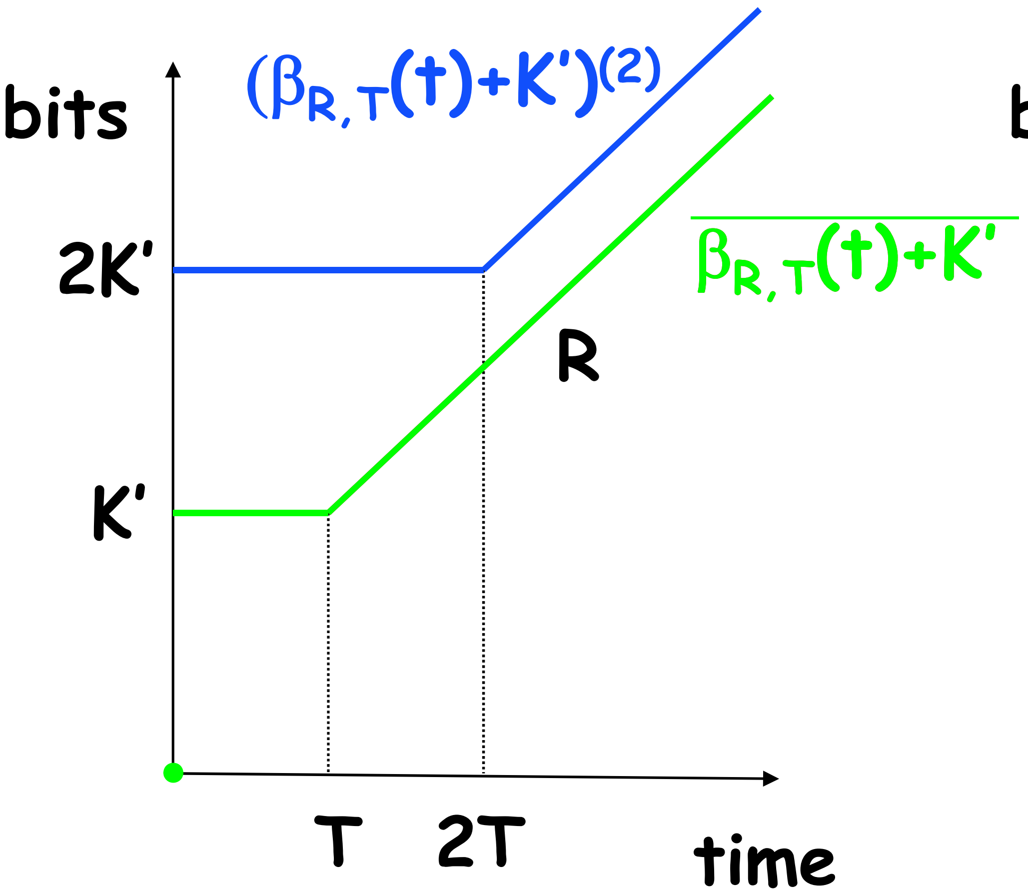




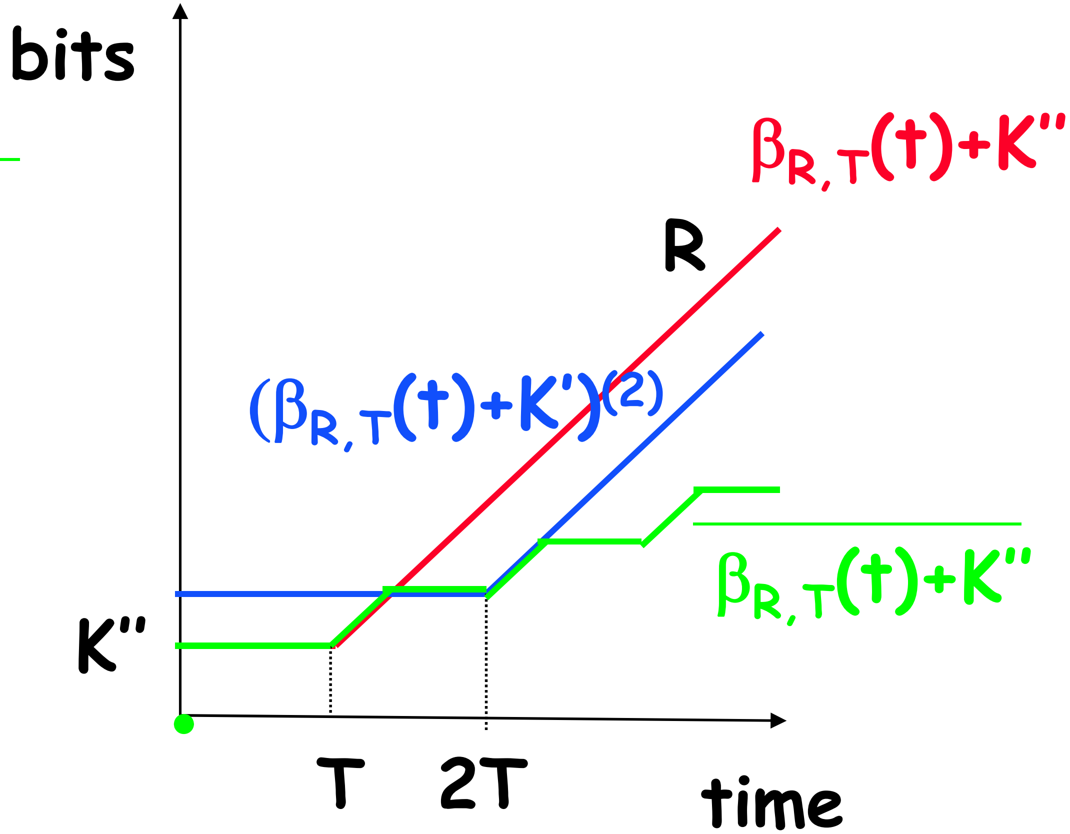
$\beta_{R,T} + K'$ is star-shaped ?
Sub-additive ?

$\beta_{R,T} + K''$ is star-shaped ?
Sub-additive ?

Examples

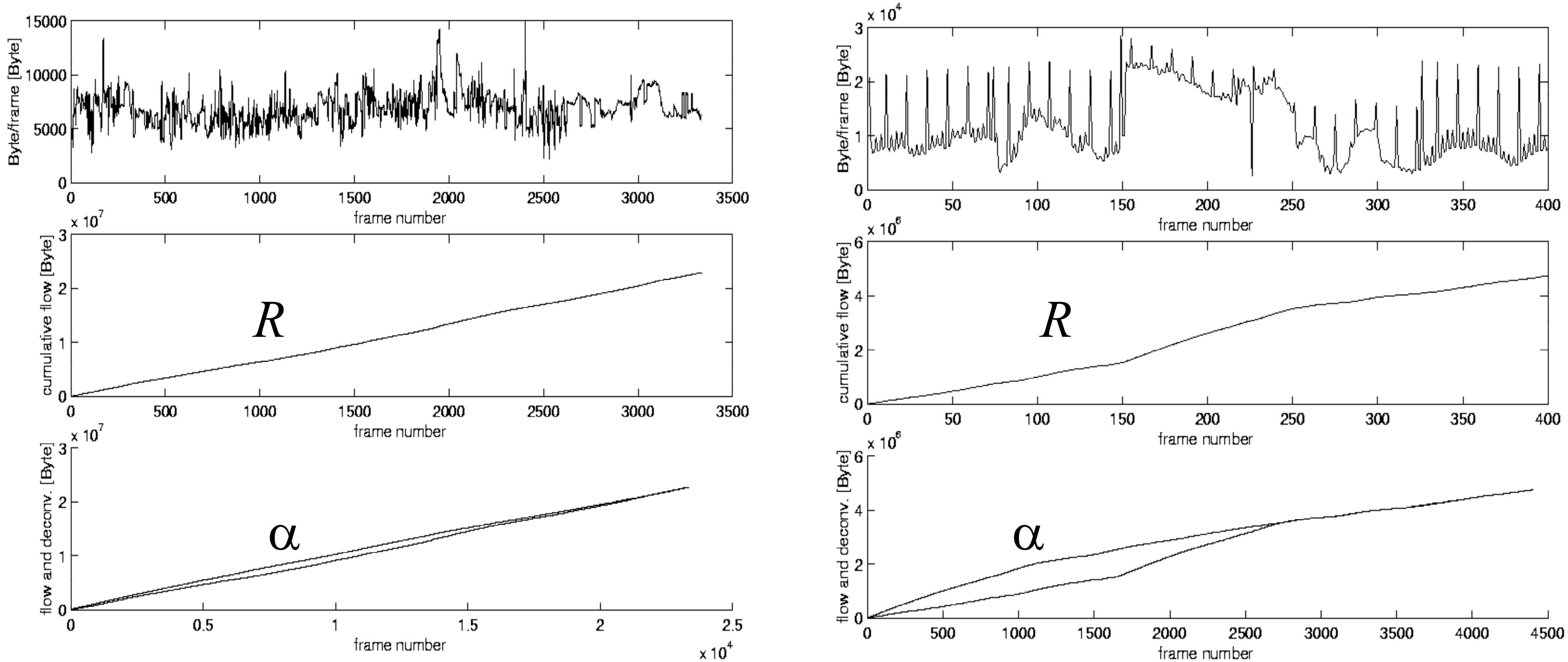


bits
$(\beta_{R,T}(t)+K')^{(2)}$
2K'
$\overline{\beta_{R,T}(t)+K'}$
R
K'
T
2T
time



bits
$\beta_{R,T}(t)+K''$
R
$(\beta_{R,T}(t)+K')^{(2)}$
$\overline{\beta_{R,T}(t)+K''}$
K''
T
2T
time

Minimum Arrival Curves



Day 2

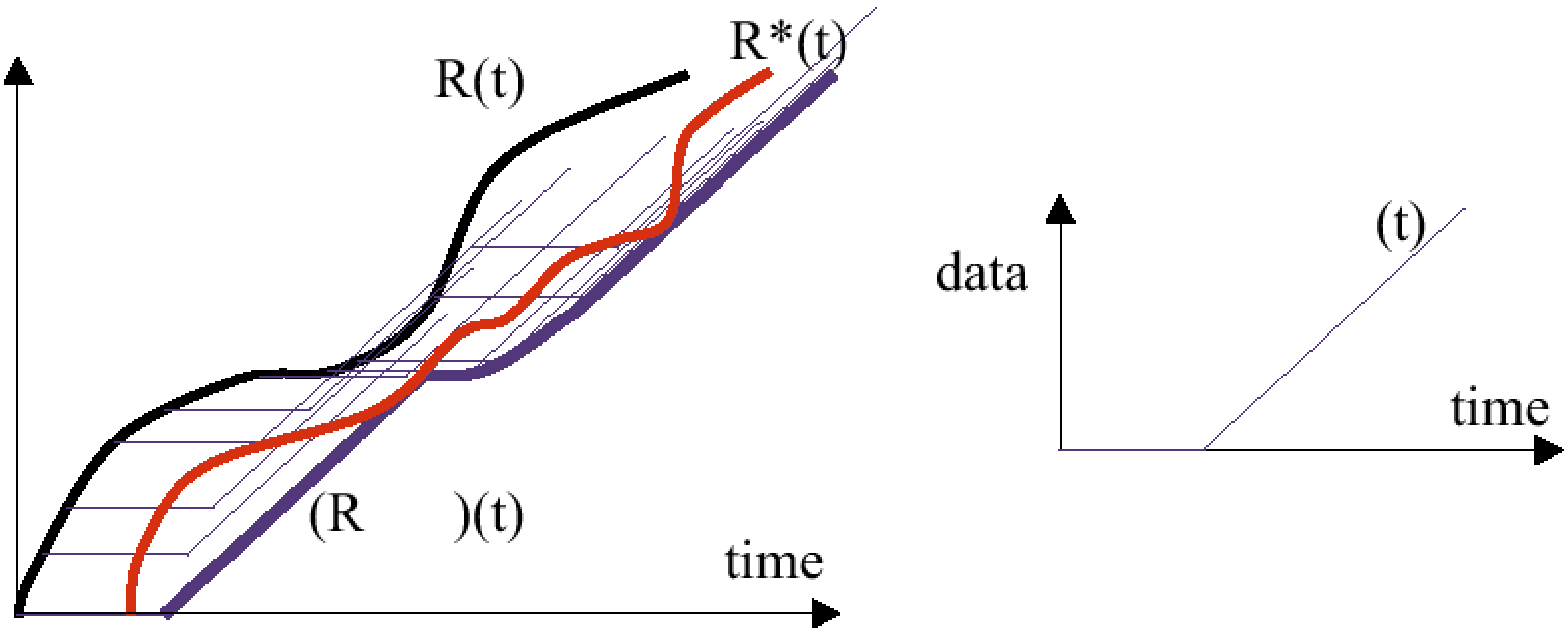


Figure 1.8: Definition of service curve. The output R^* must be above $R \otimes \beta$, which is the lower envelope of all curves $t \mapsto R(t_0) + \beta(t - t_0)$.

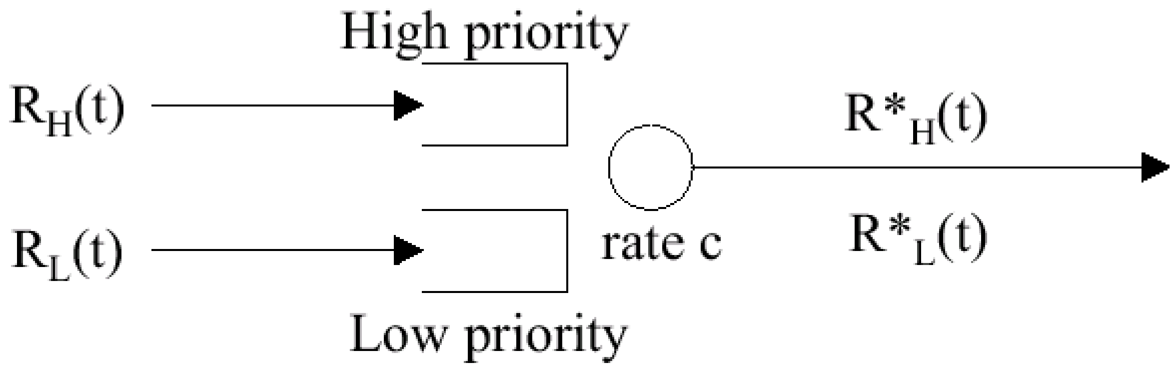


Figure 1.9: Two priority flows (H and L) served with a preemptive head of the line (HOL) service discipline. The high priority flow is constrained by arrival curve α.

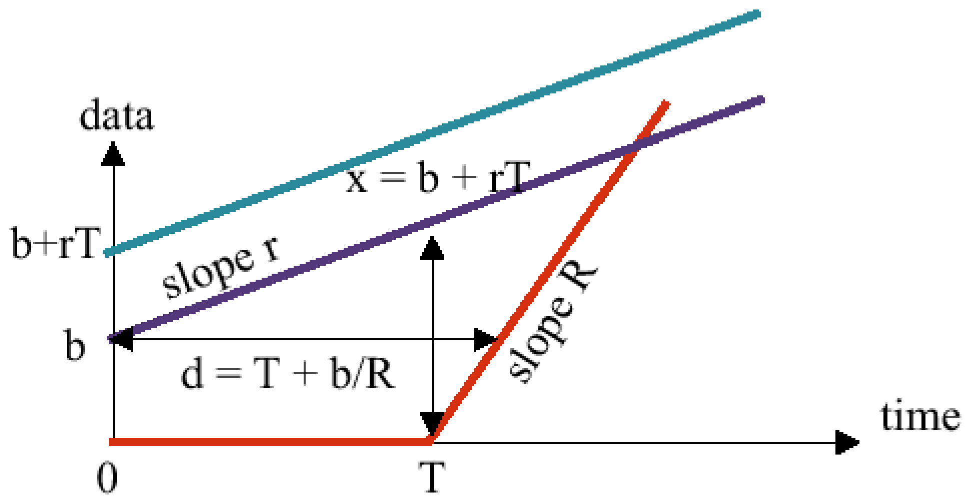


Figure 1.10: Computation of buffer, delay and output bounds for an input flow constrained by one leaky bucket, served in one node offered a rate-latency service curve. If $r \leq R$, then the buffer bound is $x = b + rT$, the delay bound is $d = T + \frac{b}{R}$ and the burstiness of the flow is increased by rT. If $r > R$, the bounds are infinite.

Proposition 1.4.1 (Intserv model, buffer and delay bounds). *Consider a VBR flow, with TSPEC (M, p, r, b), served in a node that guarantees to the flow a service curve equal to the rate-latency function $\beta = \beta_{R,T}$. The buffer required for the flow is bounded by*

$$v = b + rT + \left(\frac{b-M}{p-r} - T\right)^{+} \left[(p-R)^{+} - p + r\right]$$

The maximum delay for the flow is bounded by

$$d = \frac{M + \frac{b-M}{p-r}(p-R)^{+}}{R} + T$$

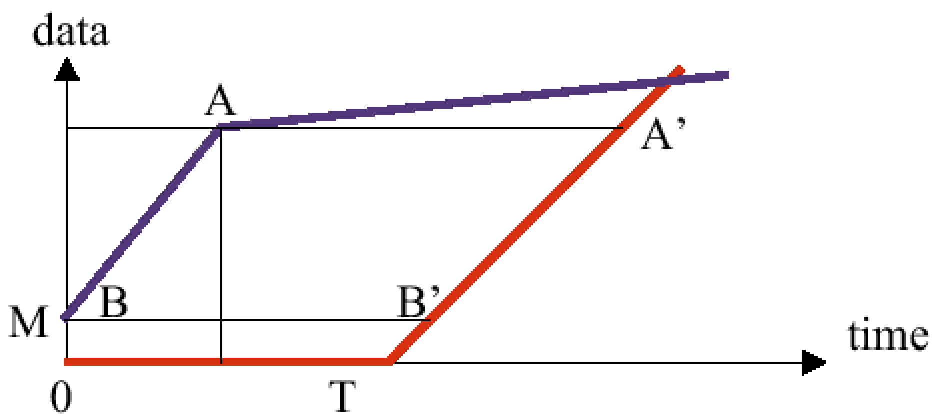
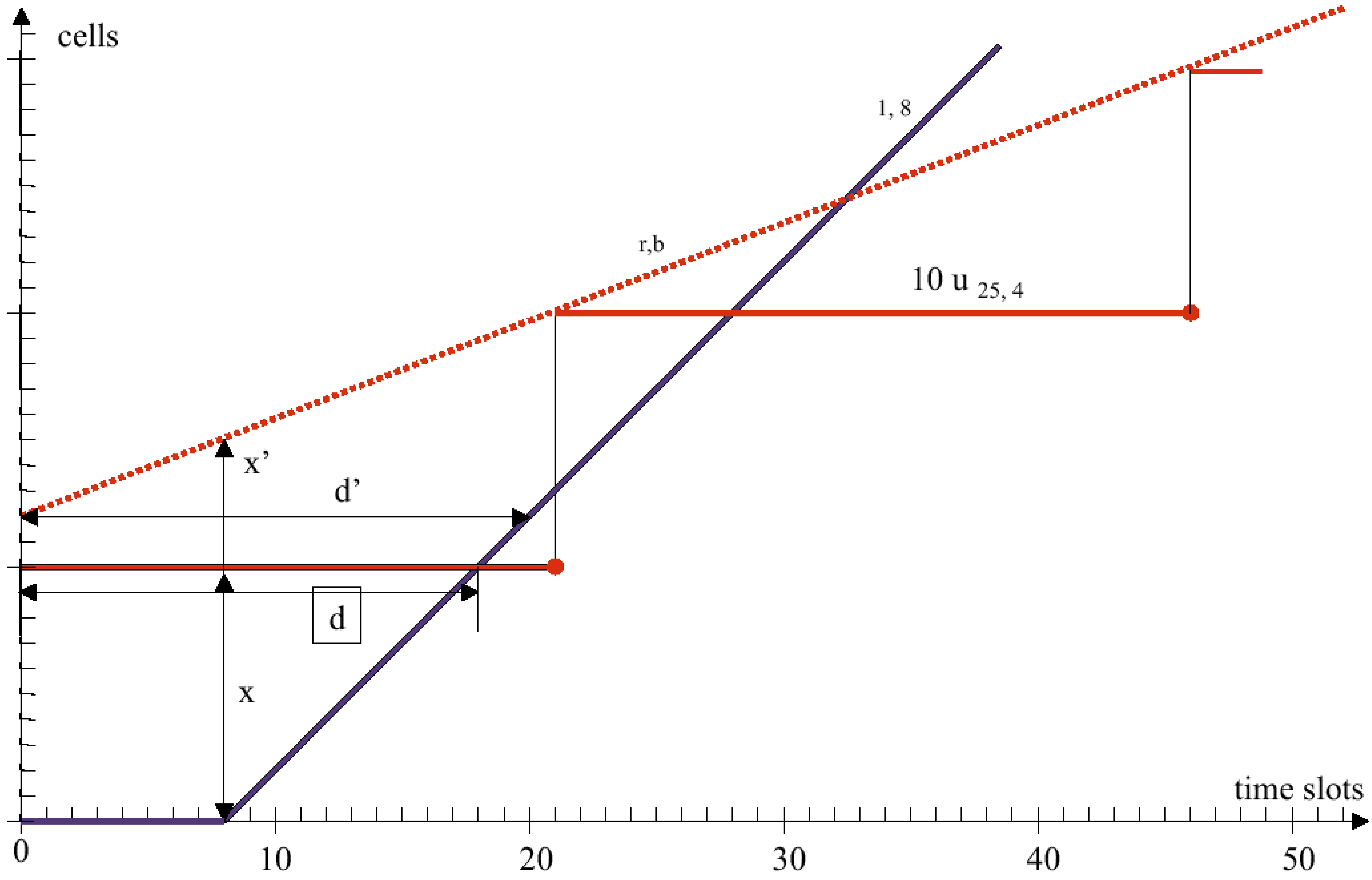
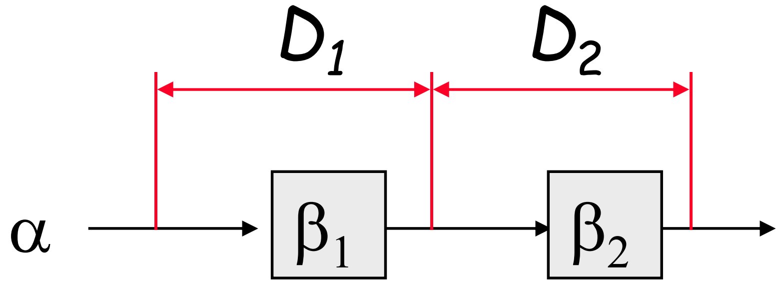


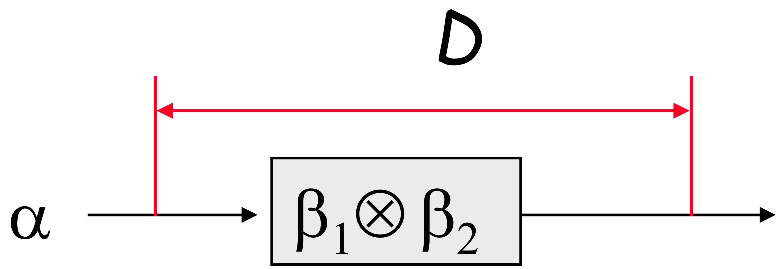
Figure 1.11: Computation of buffer and delay bound for one VBR flow served in one Intserv node.


cells
1, 8
r,b
10 u 25, 4
x'
d'
d
x
time slots
0
10
20
30
40
50

Pay Bursts Only Once



$$D_1 + D_2 \le (2b + rT_1)/R + T_1 + T_2$$



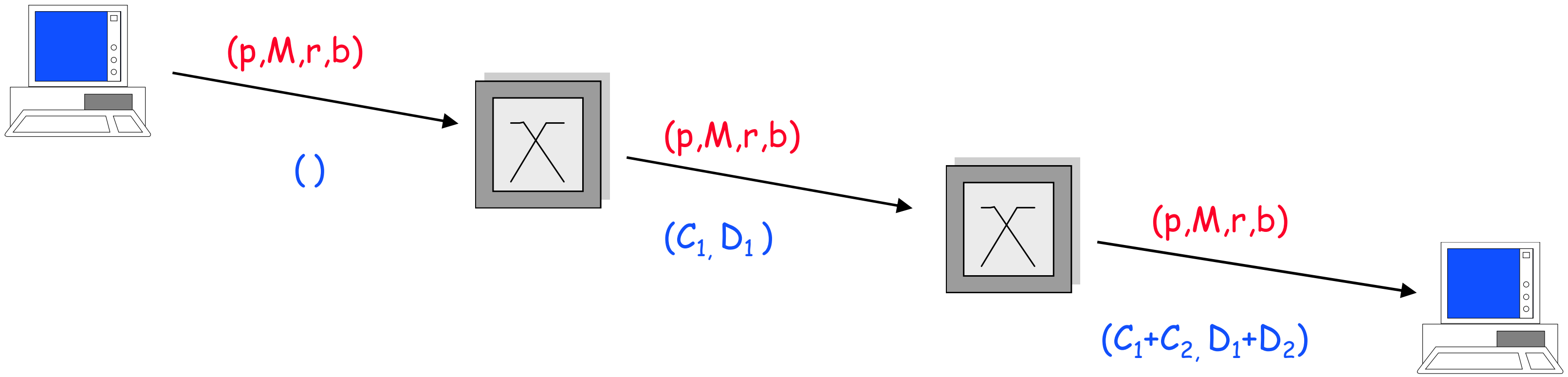
$$D \le b/R + T_1 + T_2$$

end to end delay bound is less

Application: Intserv (RSVP/IP)

- Intserv arrival curve $\alpha(t) = \min(pt+M, rt+b)$
- Rate-latency service curve $\beta_i(t) = \max(0, R_i(t-T_i))$ for node i, with $T_i = C_i/R_i + D_i$ where C_i, D_i are router-specific values
- End-to-end service curve is thus $\beta(t) = \max(0, R(t-T))$ with

  $R = \min_i R_i$ and $T = \Sigma_i T_i = \Sigma_i (C_i / R_i + D_i)$
- Flow set-up: advertizement with `PATH` message from source (`TSPEC` (p,M,r,b), computes `AD-SPEC` $(\Sigma_i C_i, \Sigma_i D_i)$ along the path)

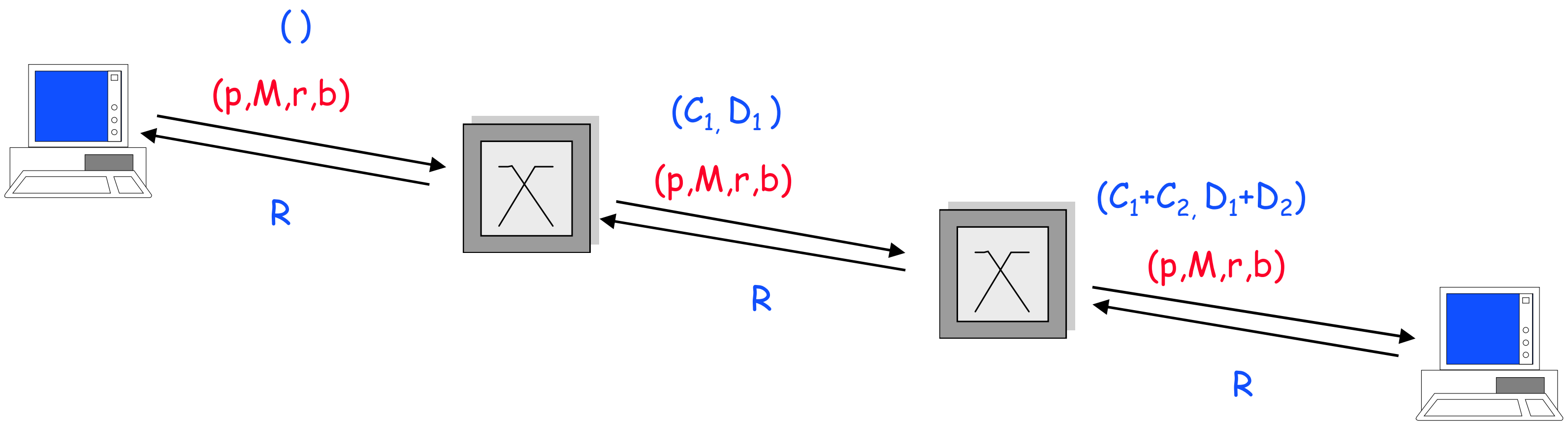


Application: Intserv (RSVP/IP)

- End-to-end delay bound is, if $R_i = R$ for all i,

  $$((b-M)/R)\ \max(0,(p-R)/(p-r)) + (M + \Sigma_i\ C_i)/R + \Sigma_i\ D_i$$

- Flow set-up: reservation with `RESV` message where R is computed so that end-to-end delay bound <= delay objective.



IntServ: Re-Shaper Buffer Dimensioning

$$B = \begin{cases} \text{if } \frac{b-M}{p-r} < T & \text{then } b + Tr \\ \text{if } \frac{b-M}{p-r} \geq T \text{ and } p > R & \text{then } M + \frac{(b-M)(p-R)}{p-r} + TR \\ \text{else} & M + Tp \end{cases}$$

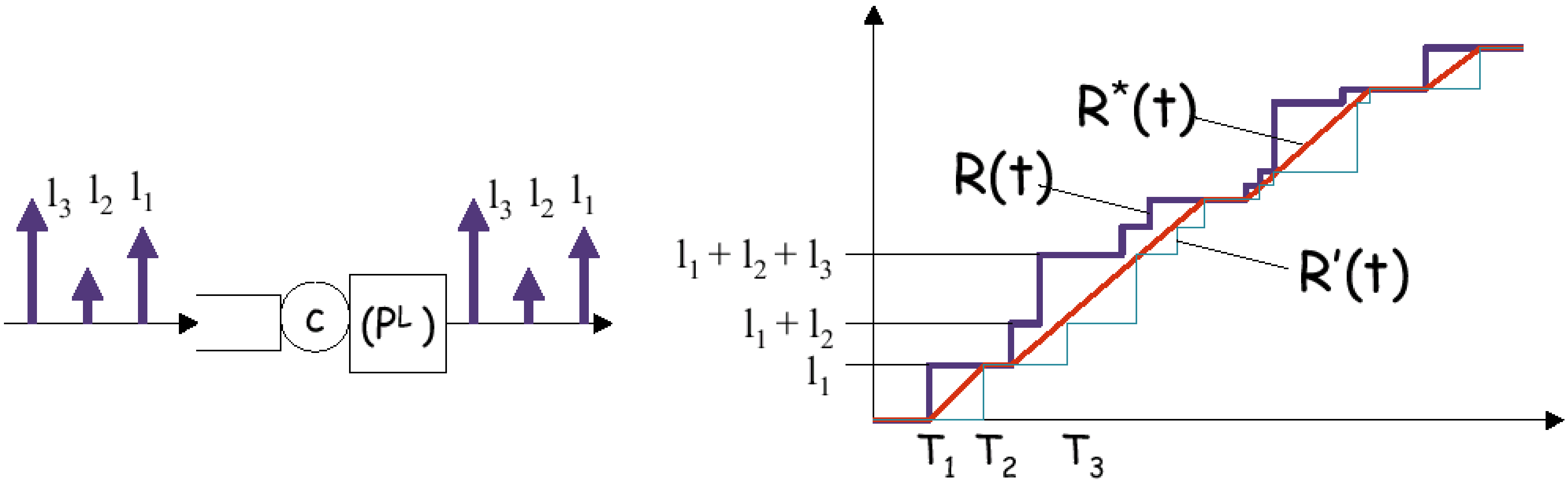


Figure 1.16: A real, variable length packet trunk of constant bit rate, viewed as the concatenation of a greedy shaper and a packetizer. The input is $R(t)$, the output of the greedy shaper is $R^*(t)$, the final output is the output of the packetizer is $R'(t)$.

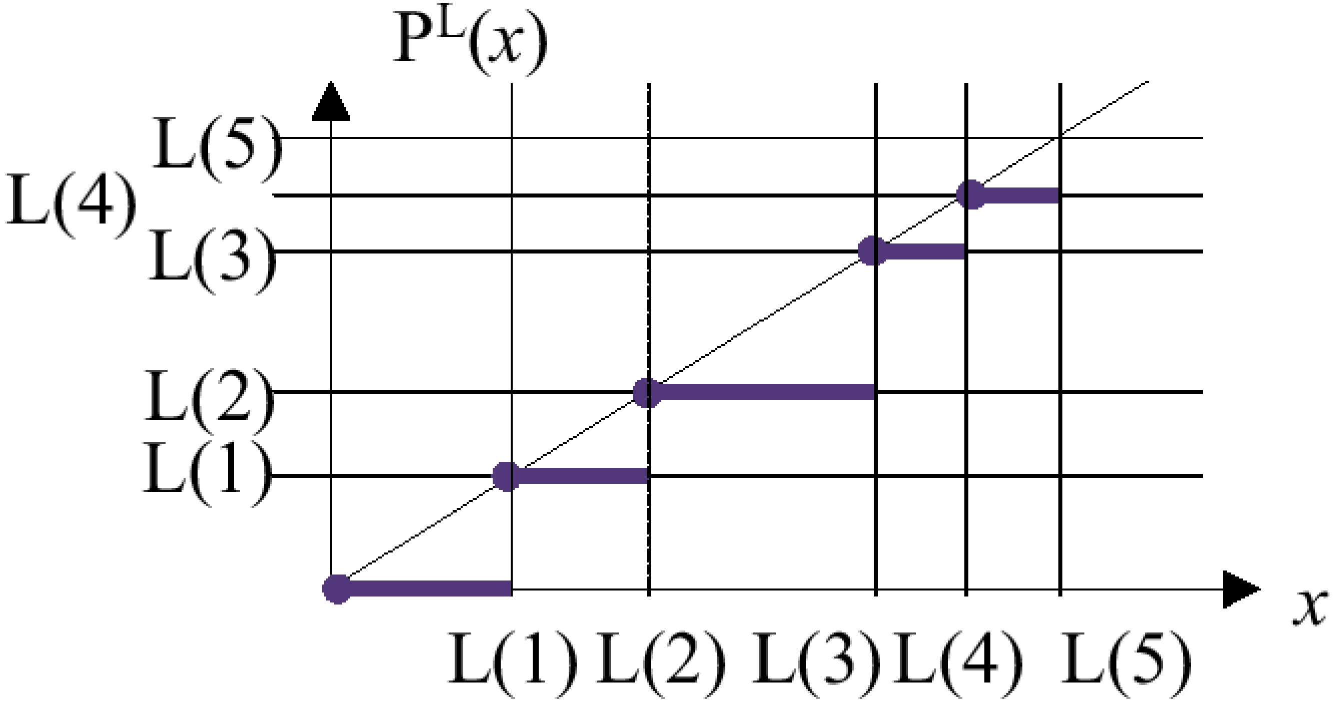
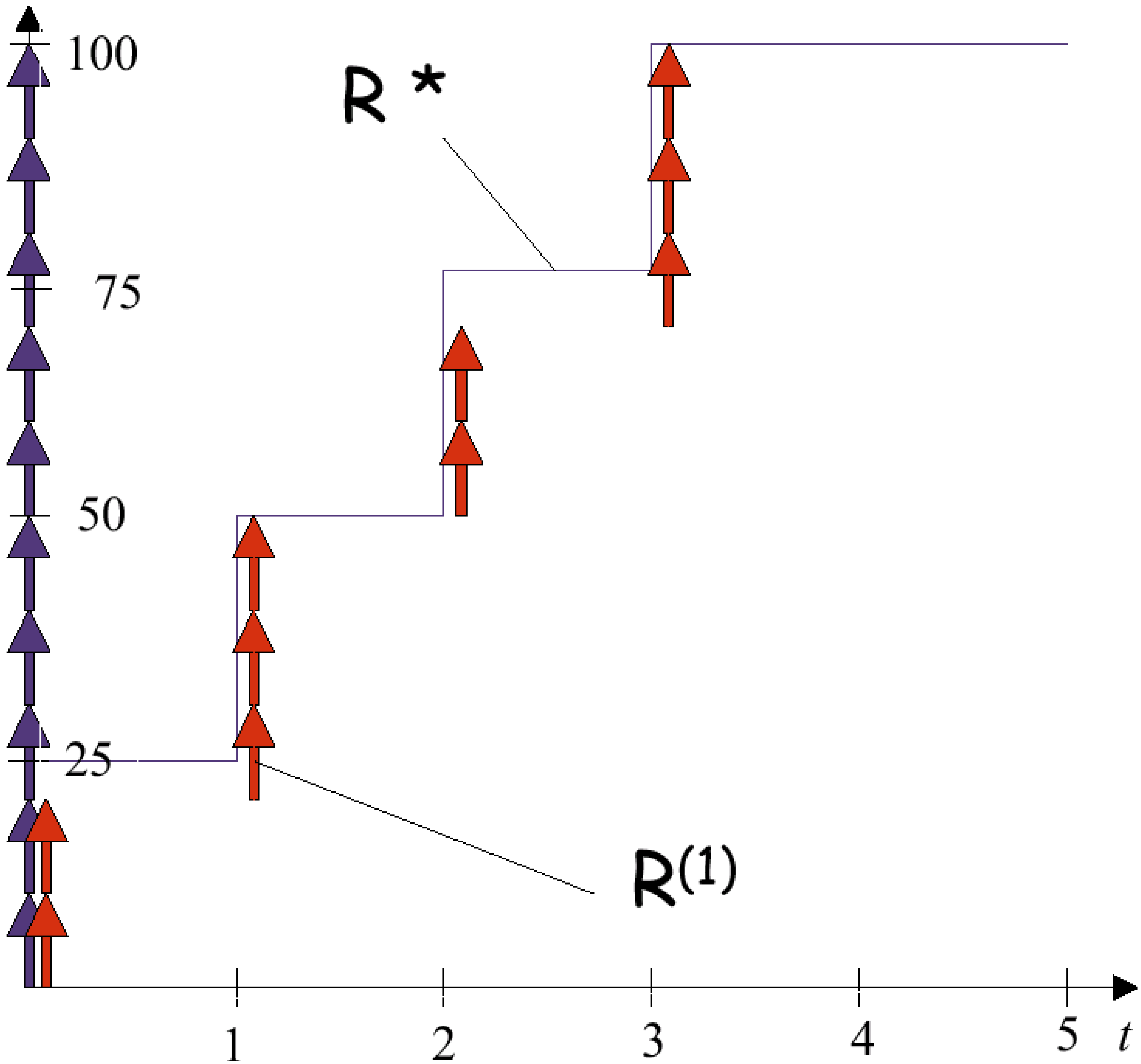
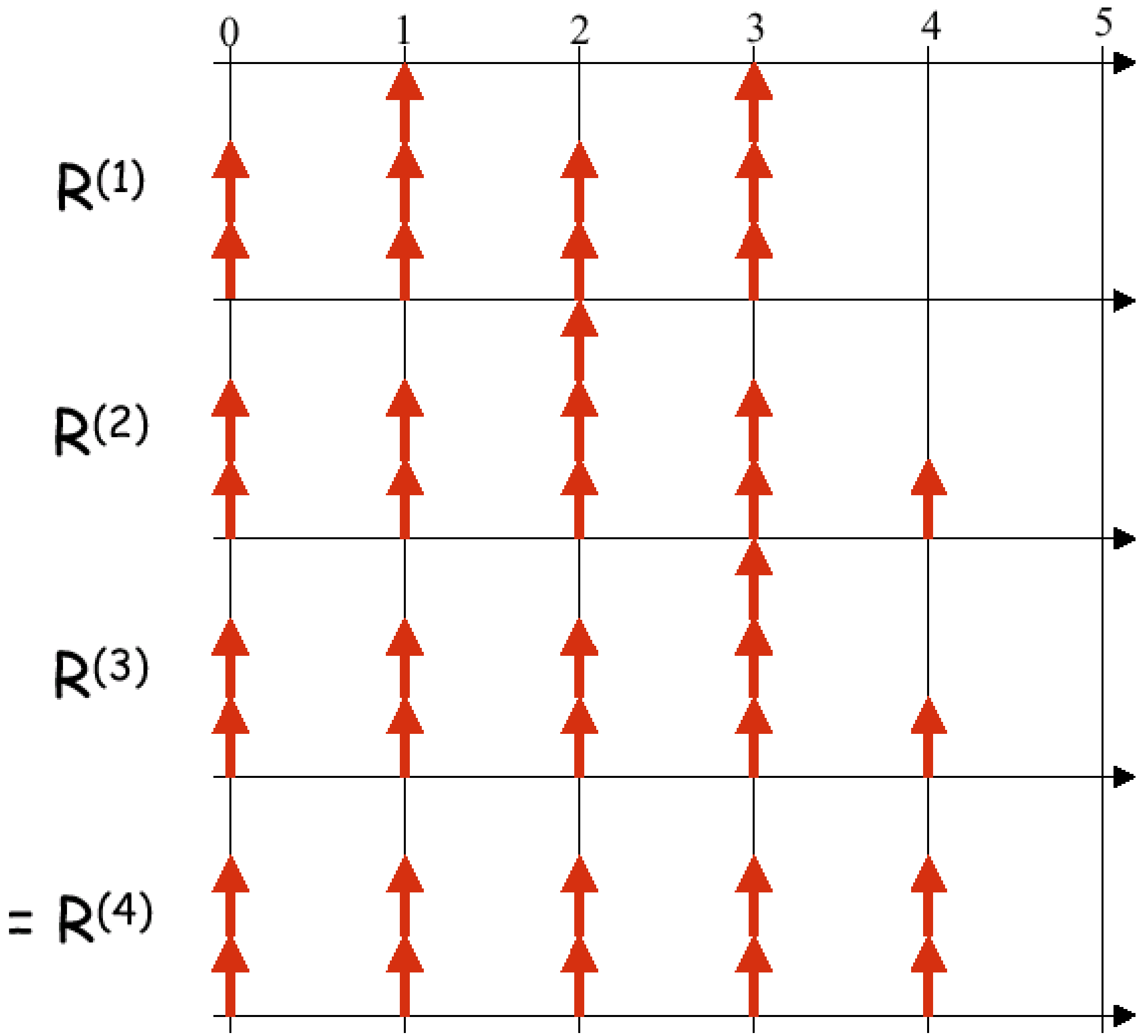
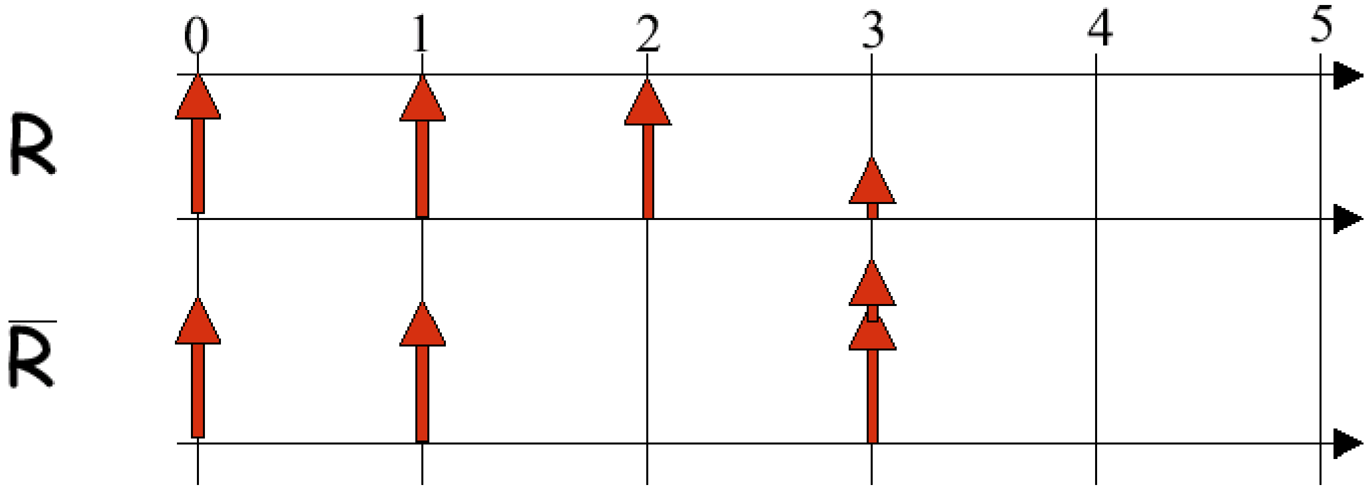


Figure 1.17: Definition of function P^L.

Day 3


100
75
50
25
R *
R(1)
1
2
3
4
5
t


0
1
2
3
4
5
R(1)
R(2)
R(3)
= R(4)



0
1
2
3
4
5
R
$\overline{R}$

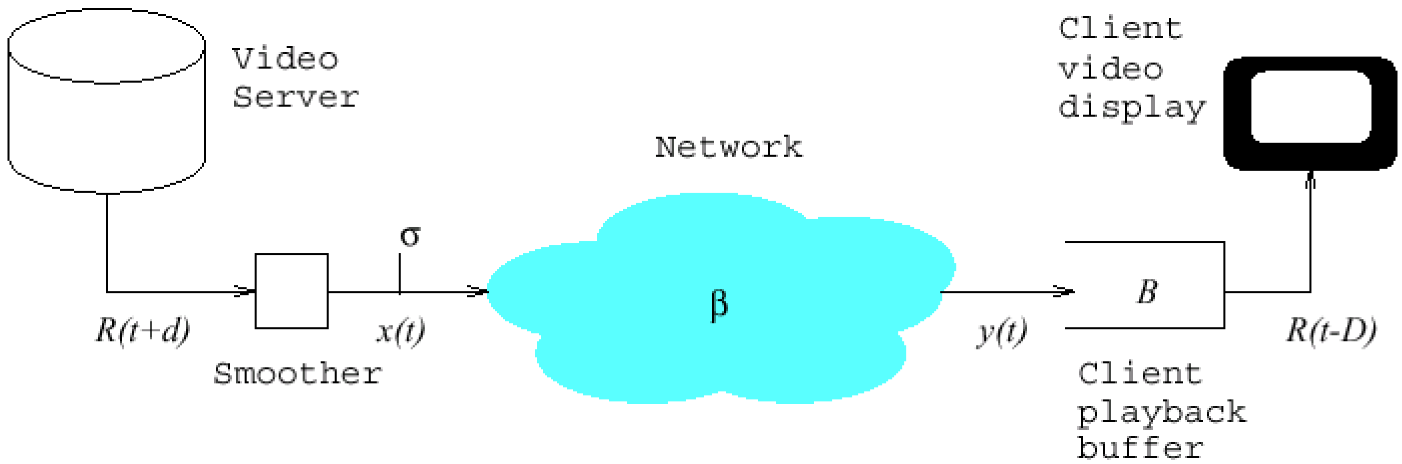
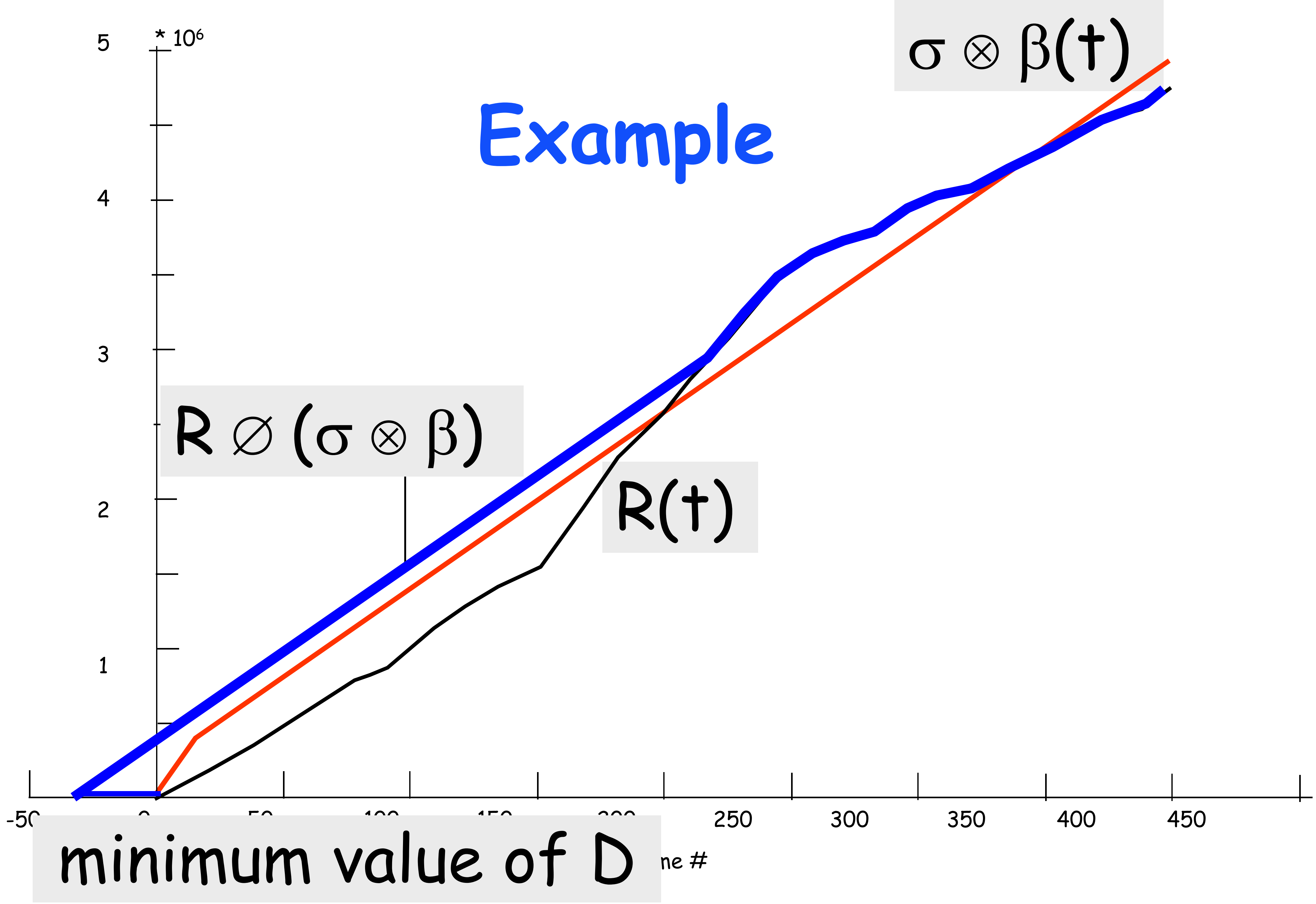


Figure 5.1: Video smoothing over a single network.

A max-plus model

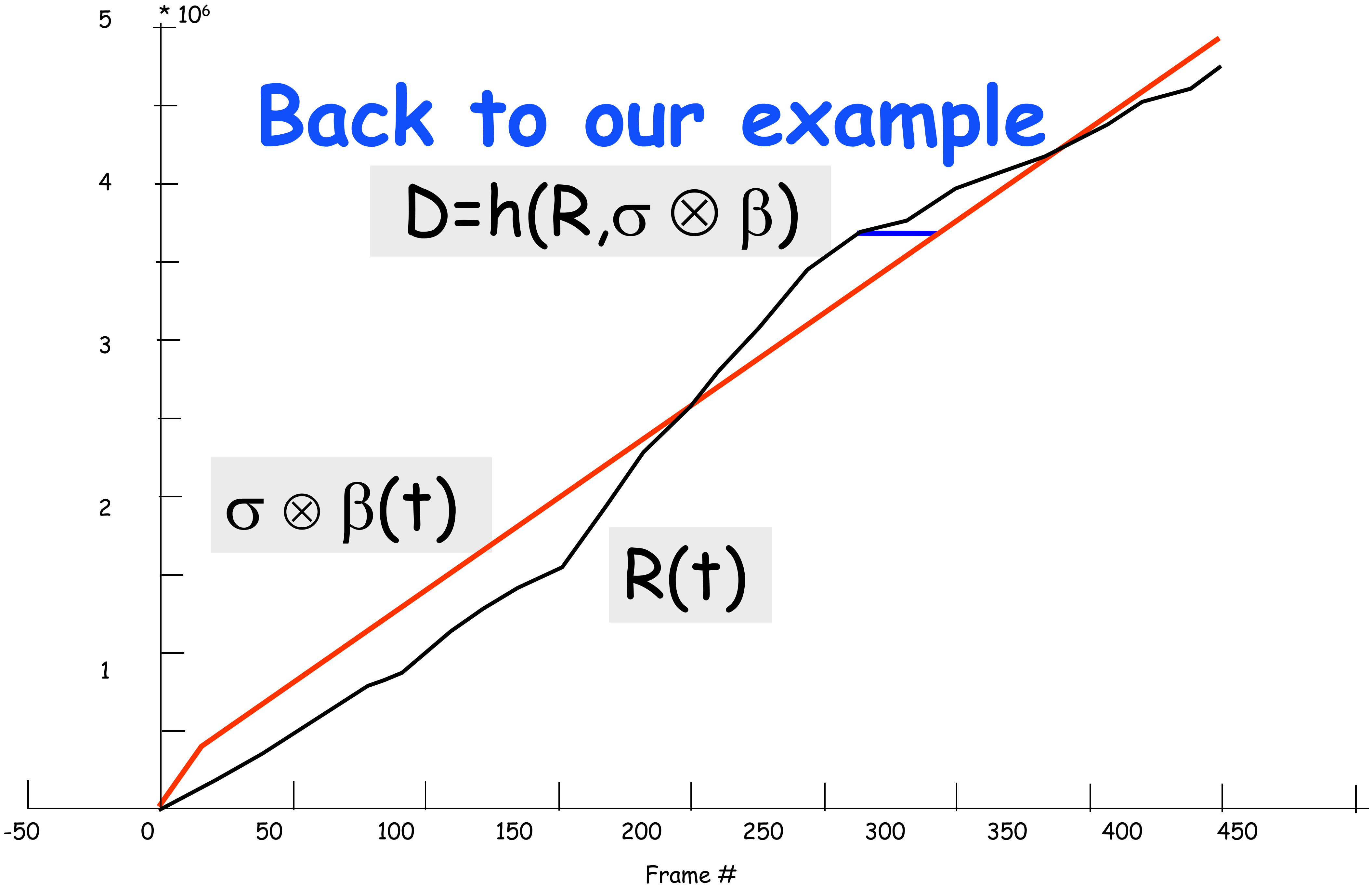
- x satisfies:
  (1) $x \geq x \oslash \sigma$
  (2) $x \geq (R \oslash \beta)(t-D)$
- a max-plus system, , with *minimum* solution
  $x^* = \inf \{x^0, x^1, \ldots, x^i, \ldots\}$
  $x^0(t) = (R \oslash \beta)(t-D)$
  $x^i = x^{i-1} \oslash \sigma$
- thus $x = (R \oslash \beta) \oslash \sigma\, (t-D) = R \oslash (\beta \otimes \sigma)\, (t-D)$

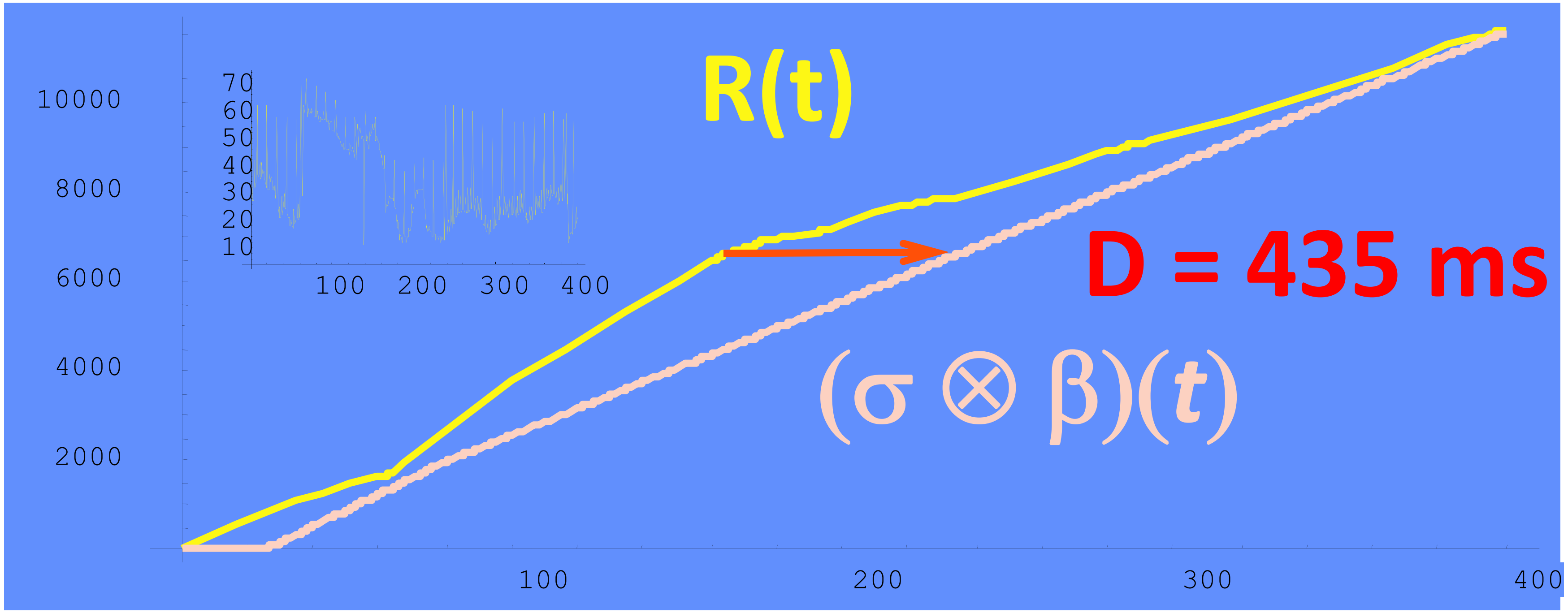
Example

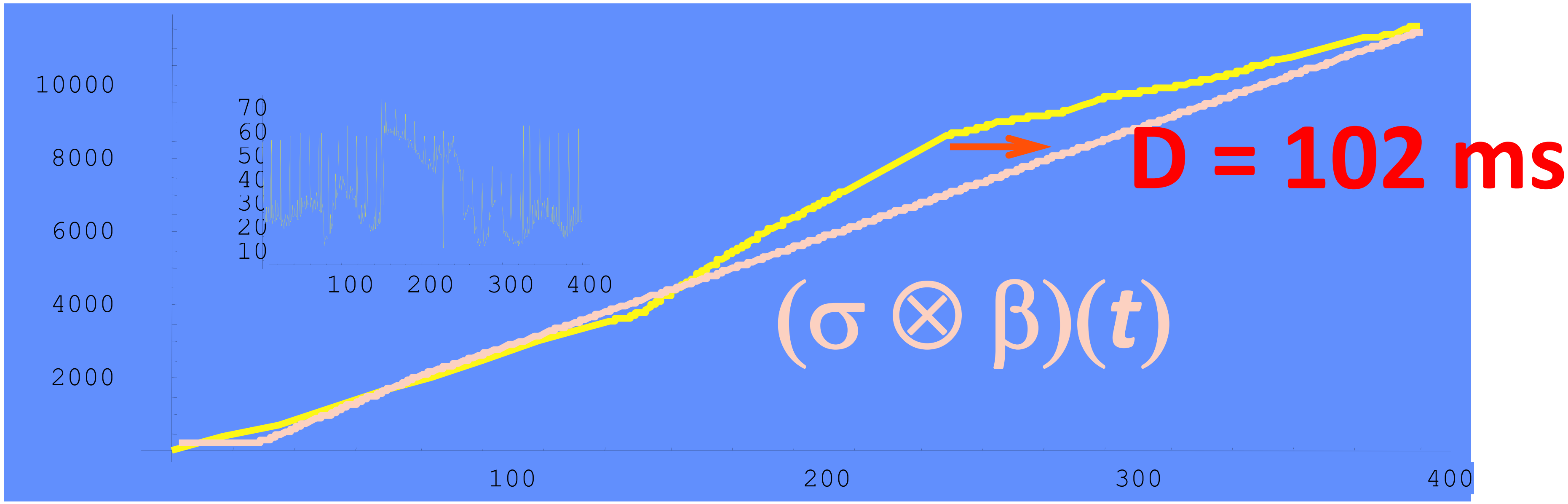


$\sigma \otimes \beta(t)$
$R \oslash (\sigma \otimes \beta)$
$R(t)$
* 10^6
5
4
3
2
1
-50
250
300
350
400
450
minimum value of D

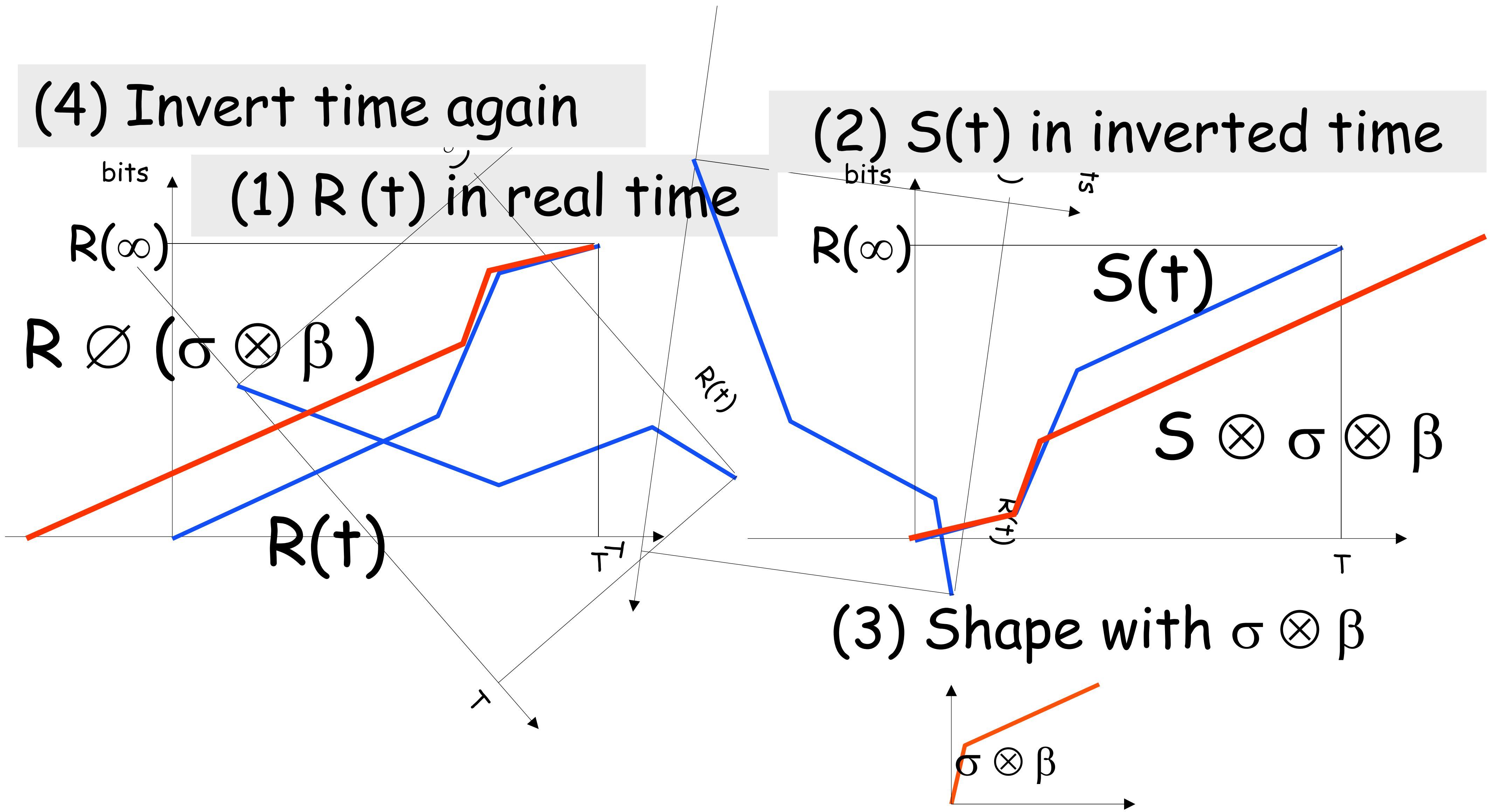
Back to our example



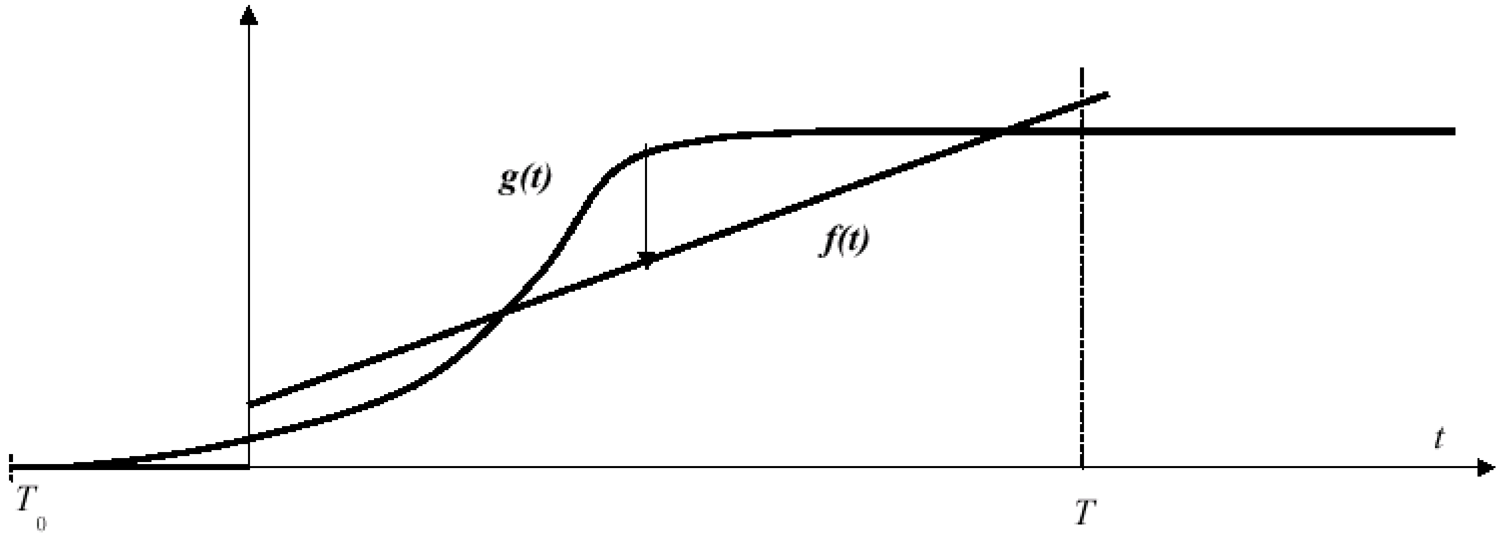

R(t)
D = 435 ms
$(\sigma \otimes \beta)(t)$


D = 102 ms
$(\sigma \otimes \beta)(t)$

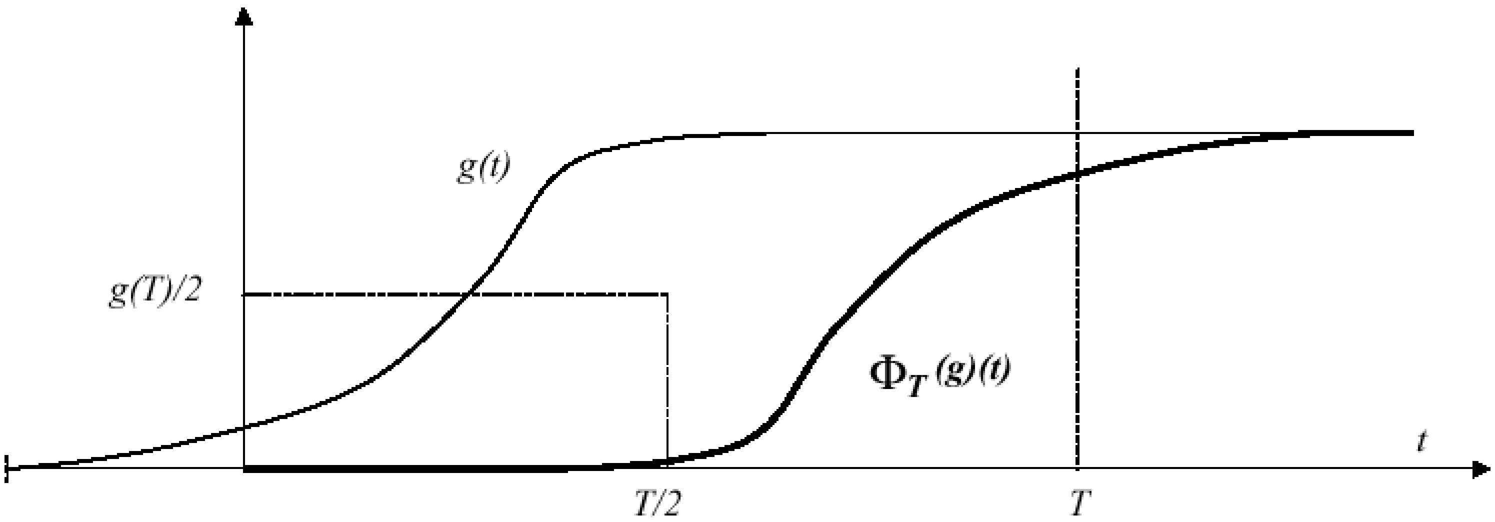
Deconvolution is the time inverse of convolution



Compute
$g \varnothing f$

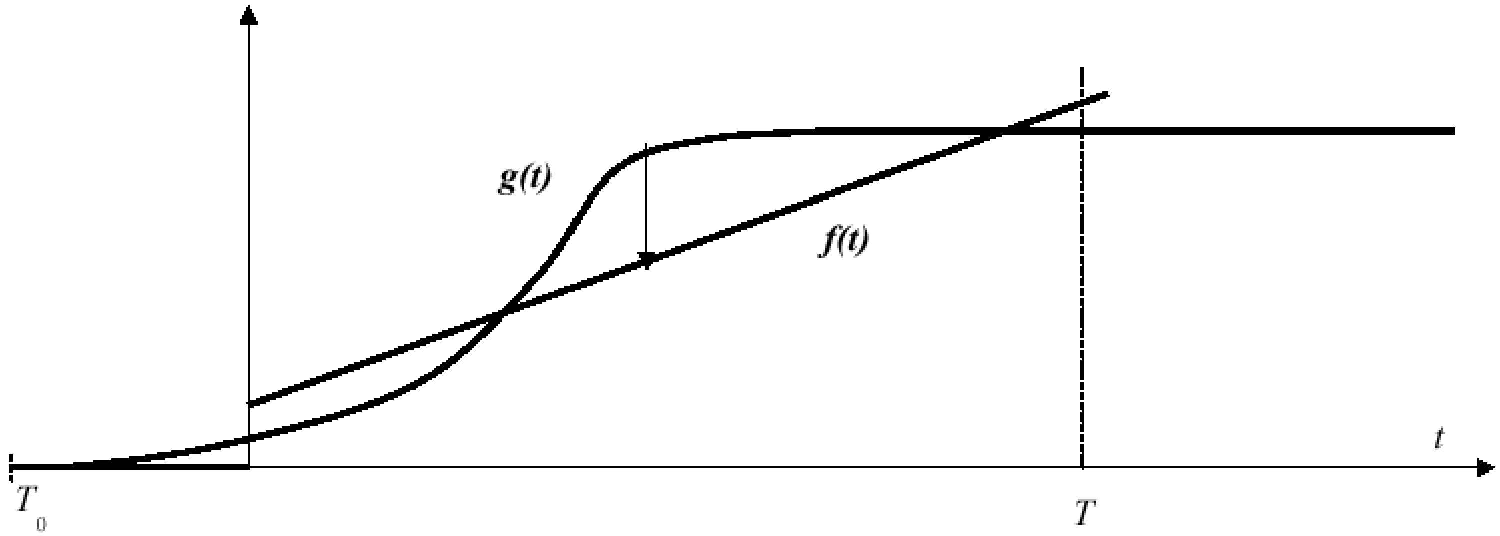


1. rotate g

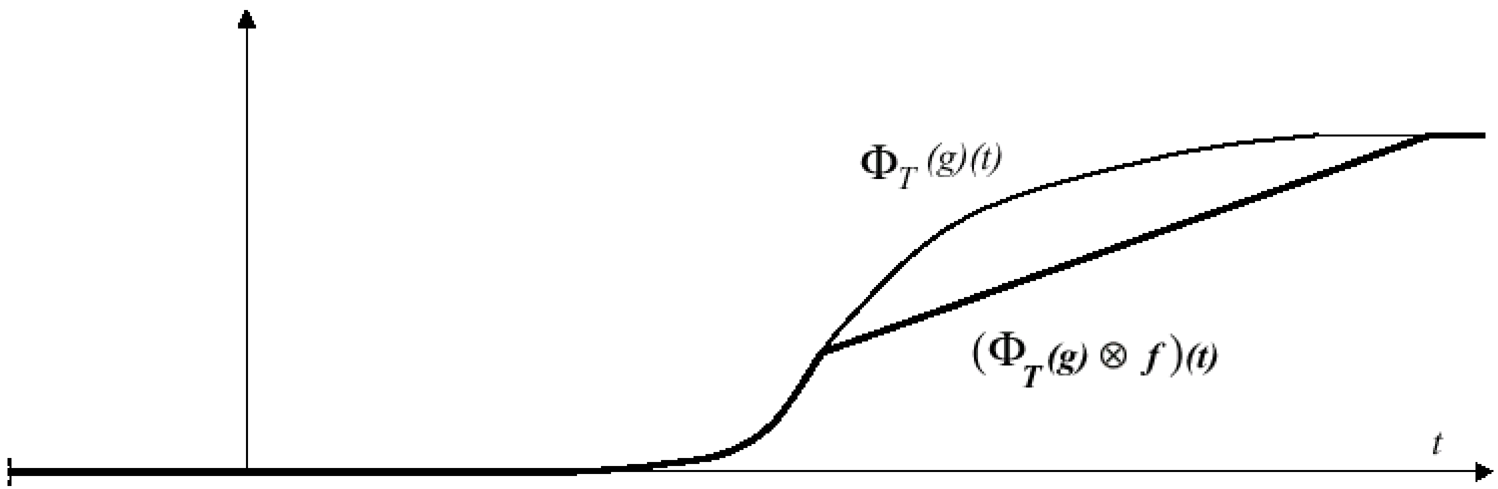


Compute

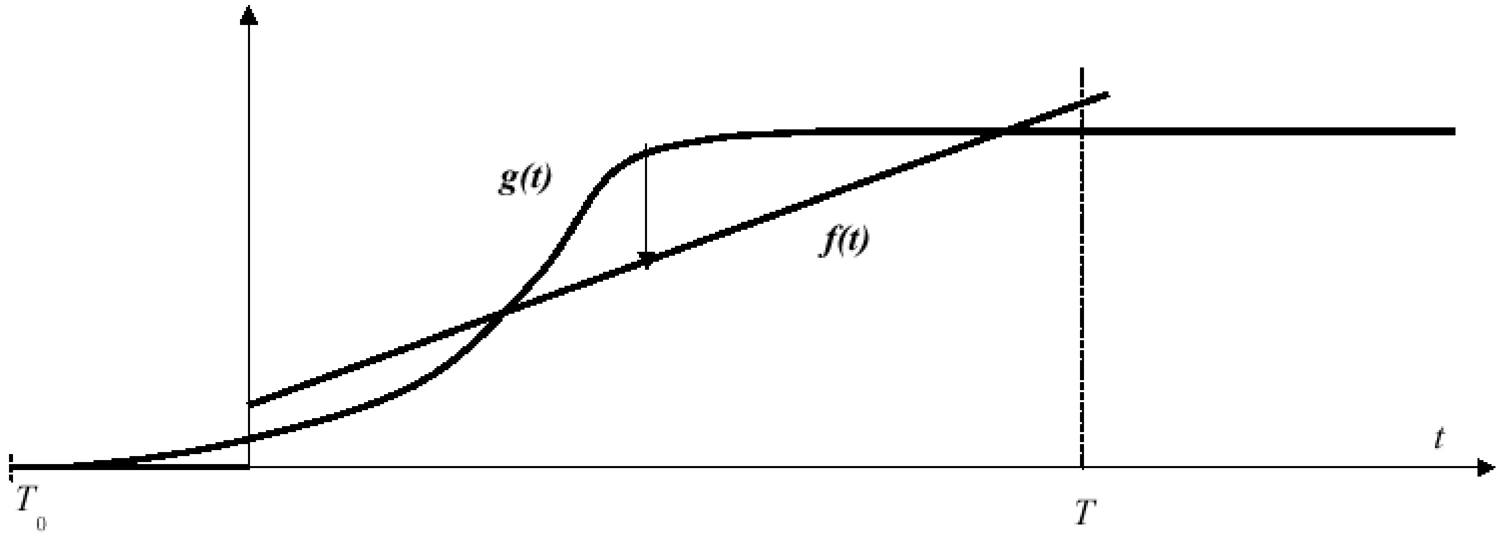
$g \oslash f$



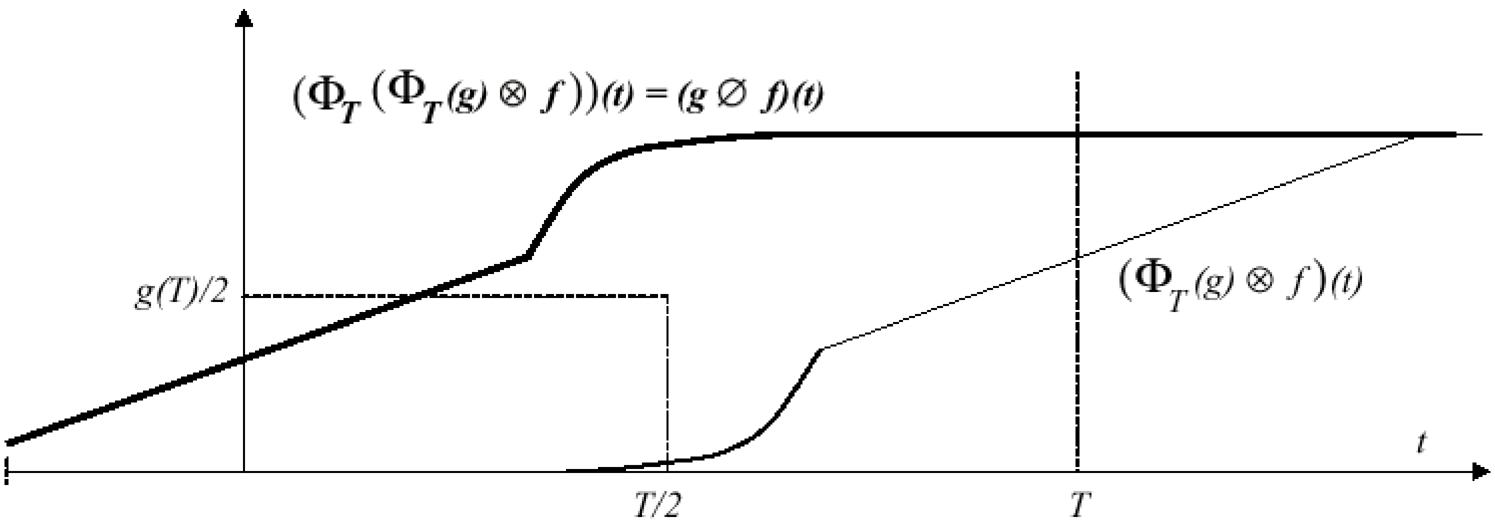
2. shape by f



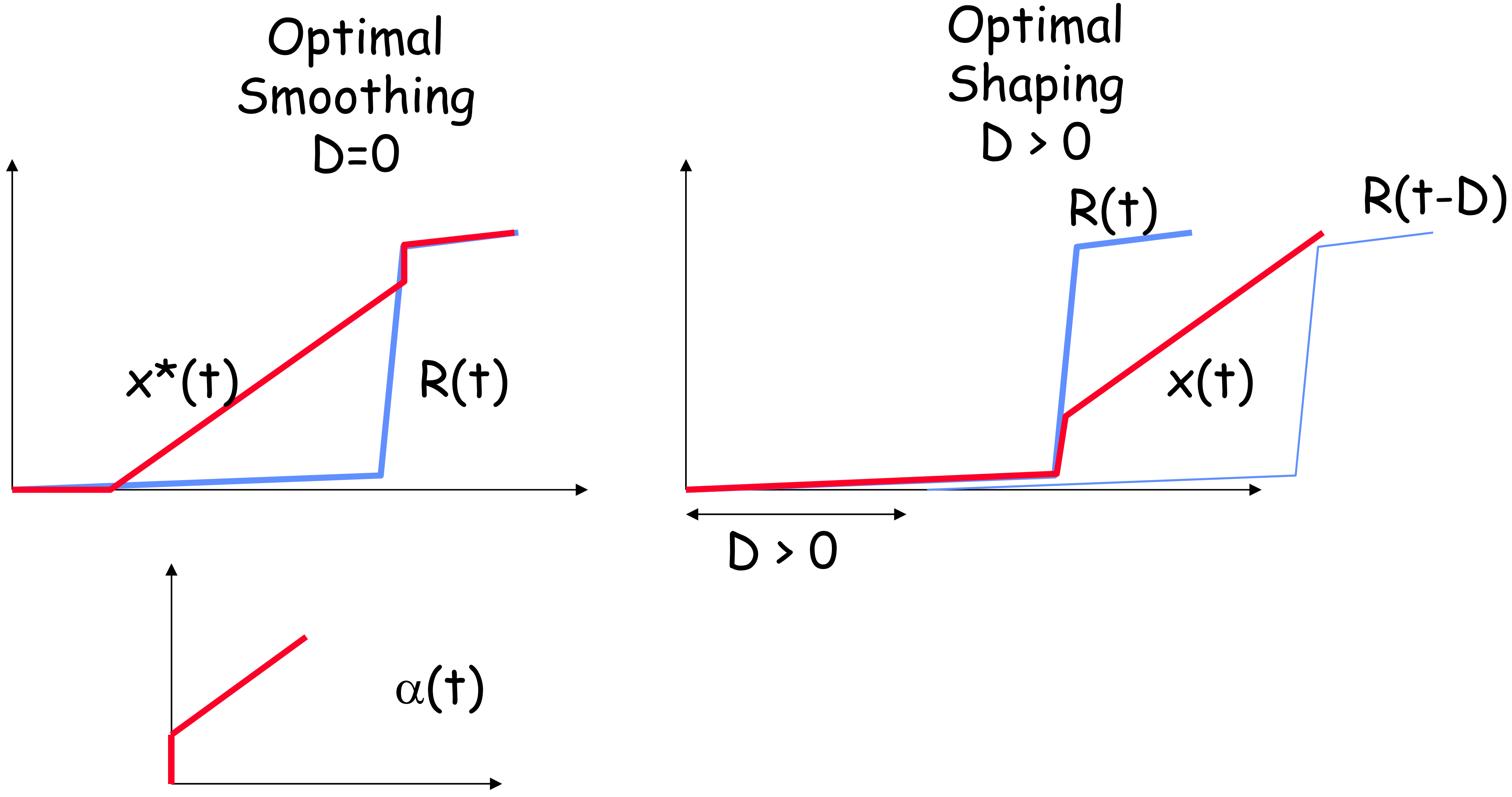
Compute
$g \oslash f$

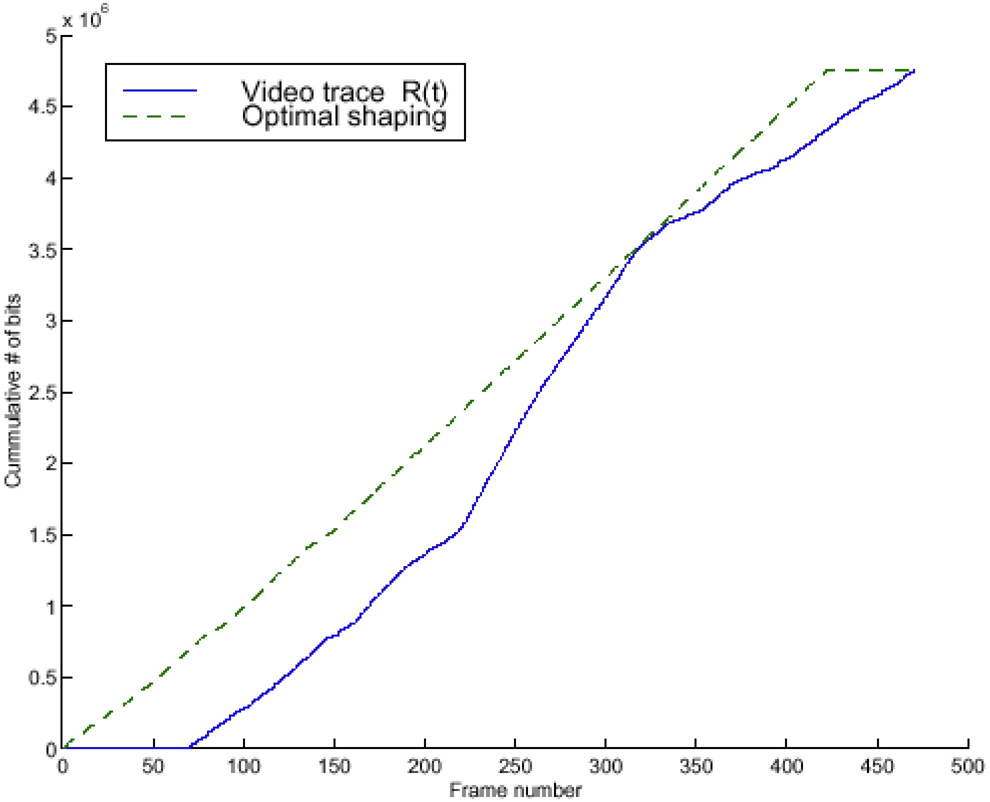


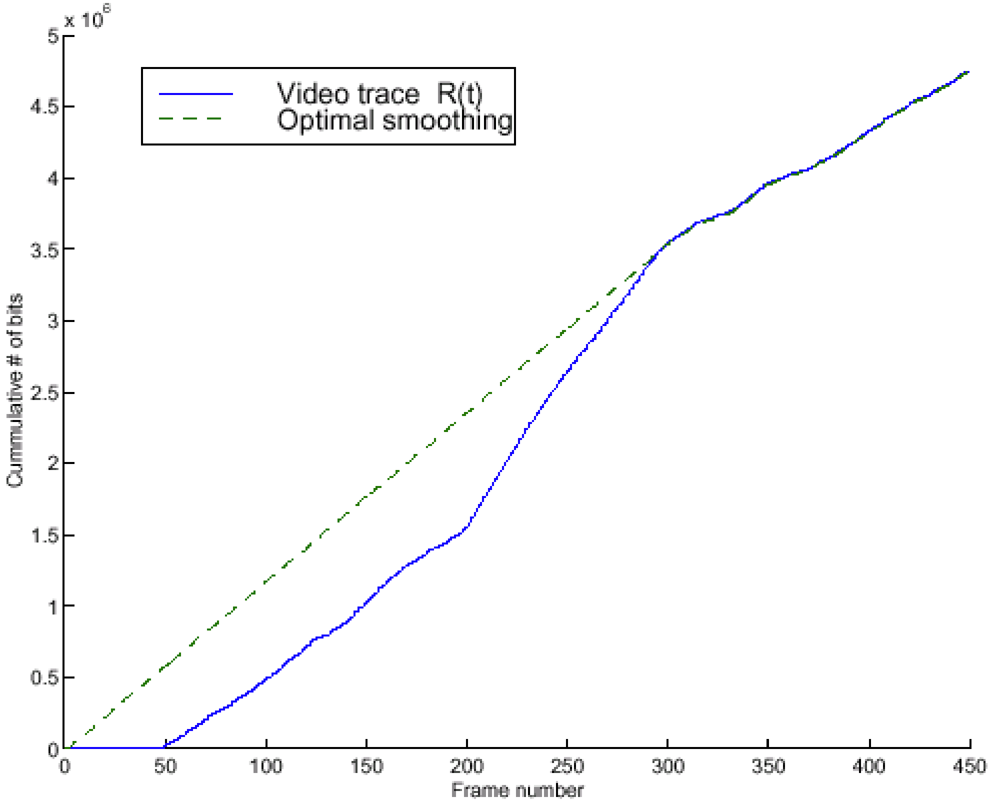
3. rotate again



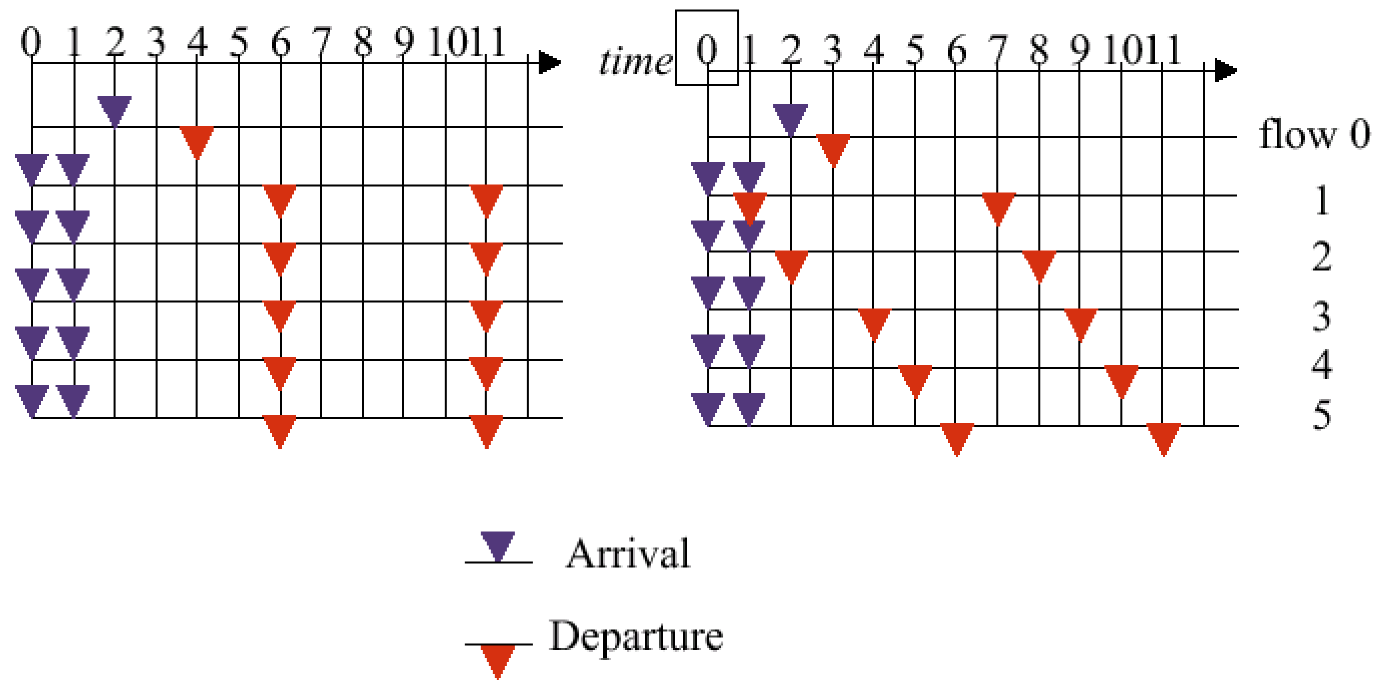
Shaping Versus Smoothing



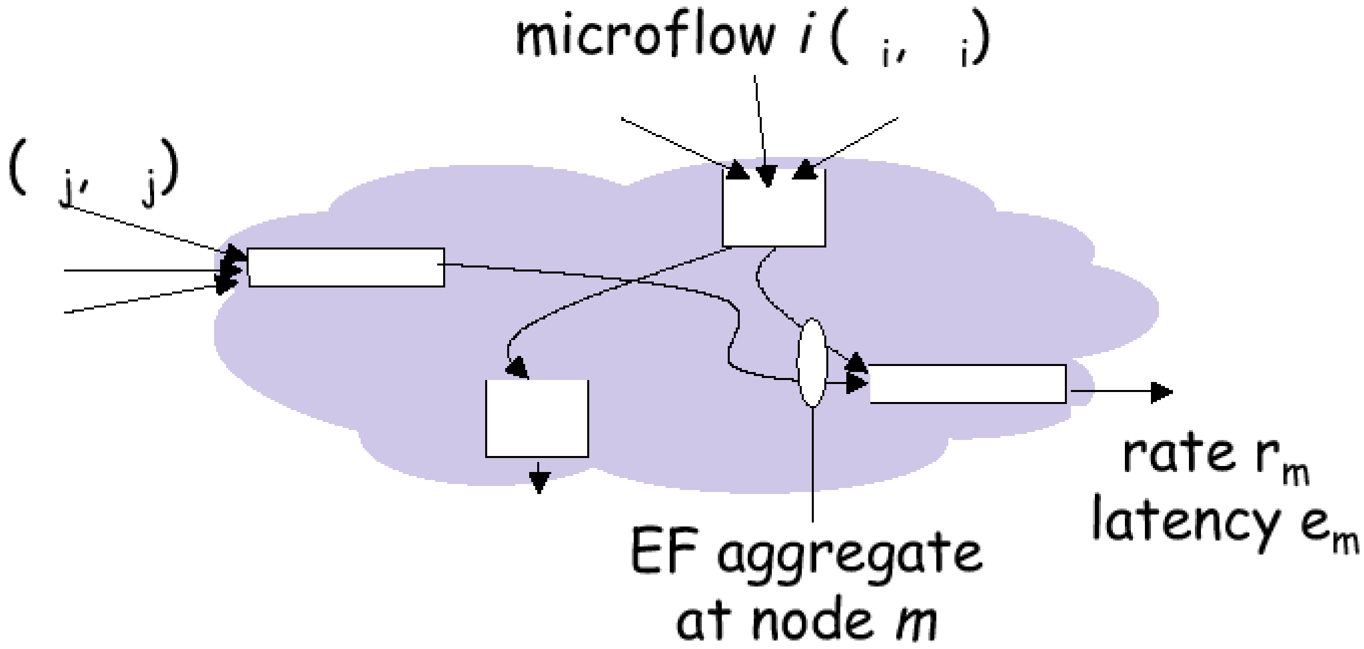

x 10^6
Video trace R(t)
Optimal shaping
Cummulative # of bits
Frame number

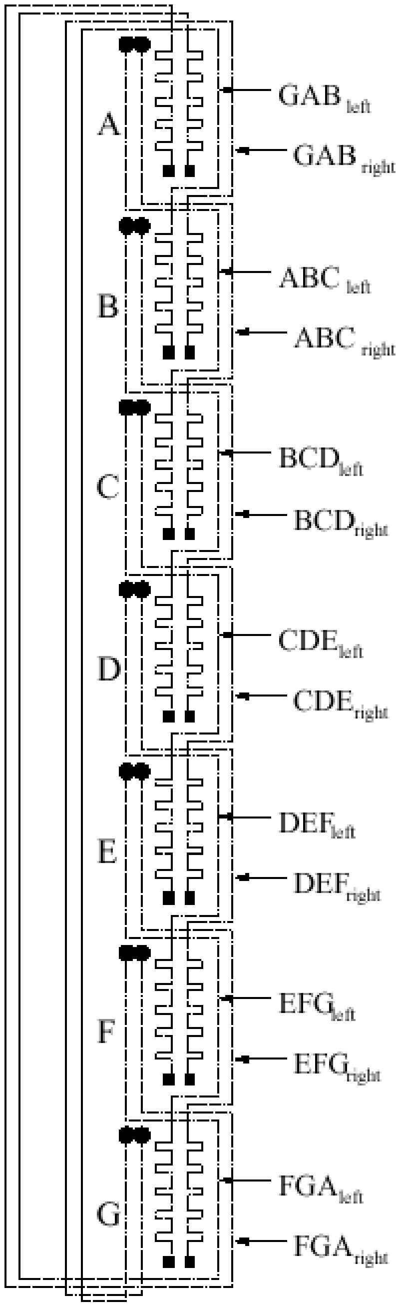

x 10^6
Video trace R(t)
Optimal smoothing
Cummulative # of bits
Frame number

Day 4



0 1 2 3 4 5 6 7 8 9 10 11
time 0 1 2 3 4 5 6 7 8 9 10 11
flow 0
1
2
3
4
5
Arrival
Departure


microflow i (i, i)
(j, j)
rate r_m
latency e_m
EF aggregate
at node m



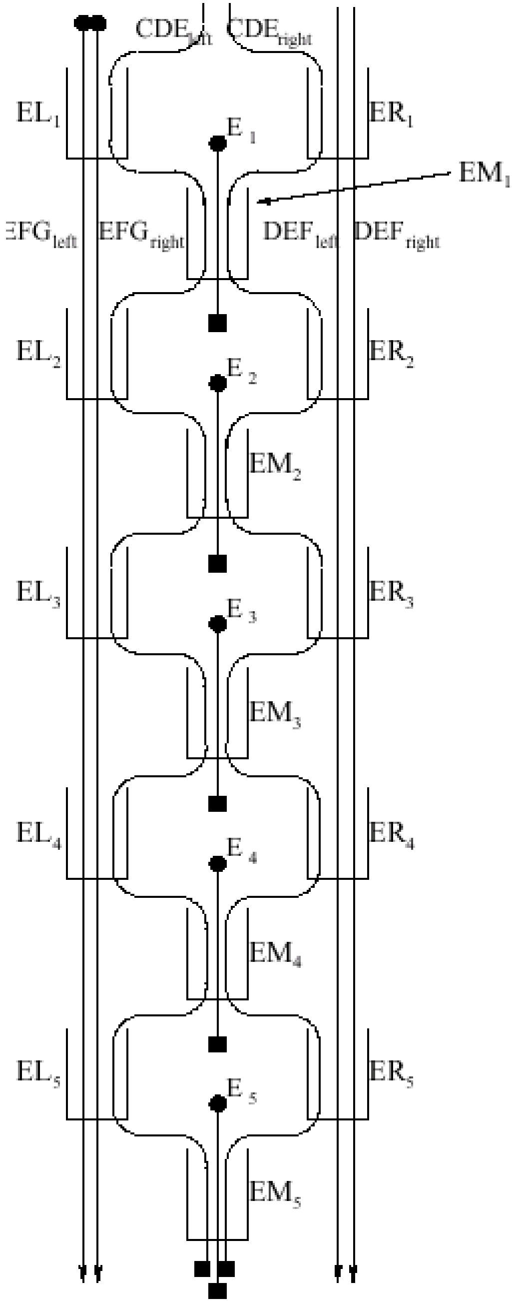


Figure 1: Subnetwork E.

« Instability of FIFO in Session-Oriented Networks »
Matthew Andrews
Bell Labs
SODA 2000

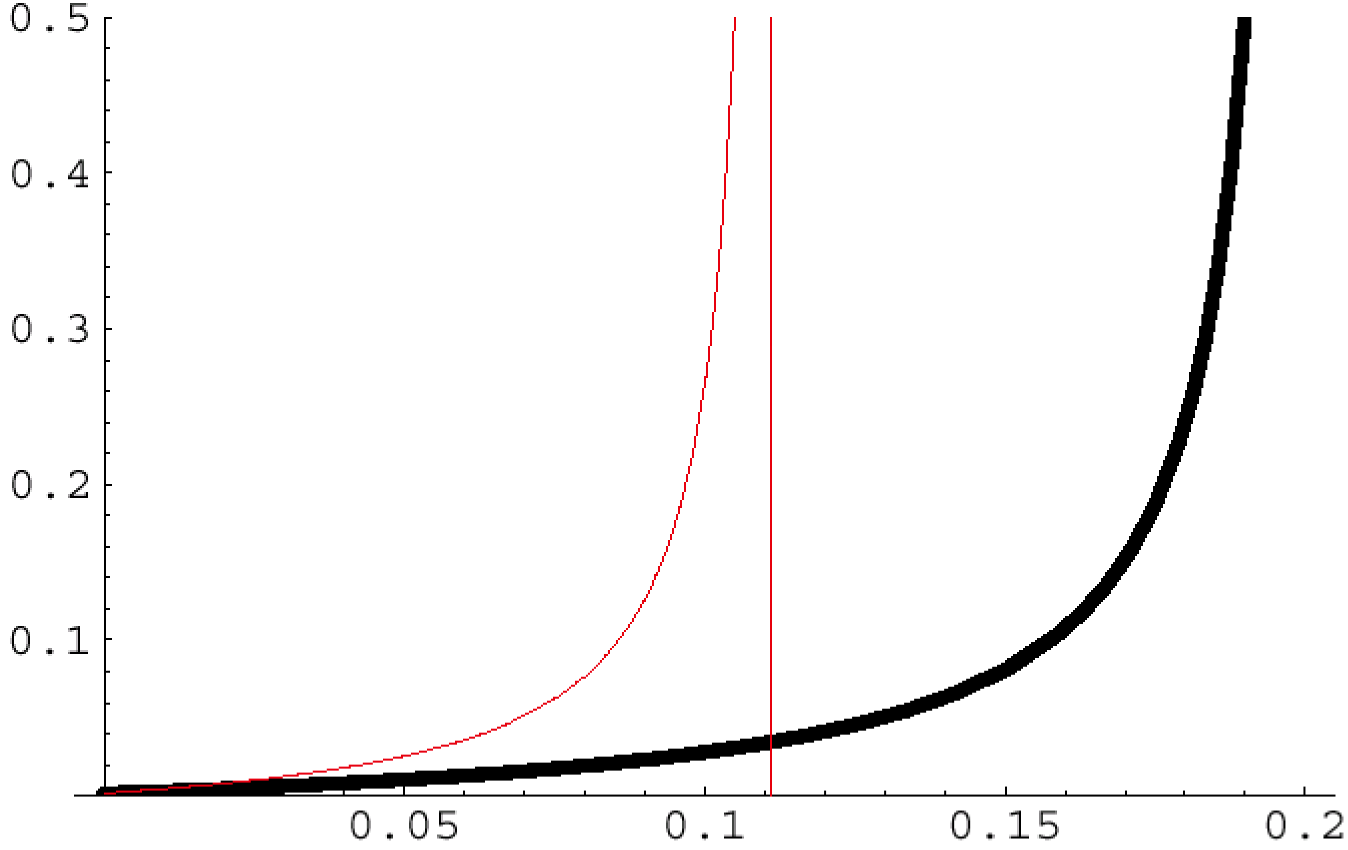
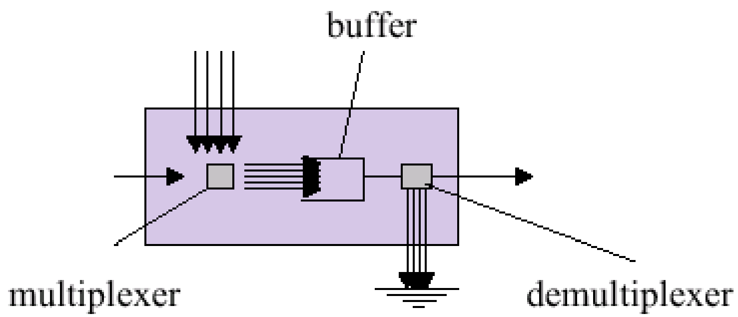
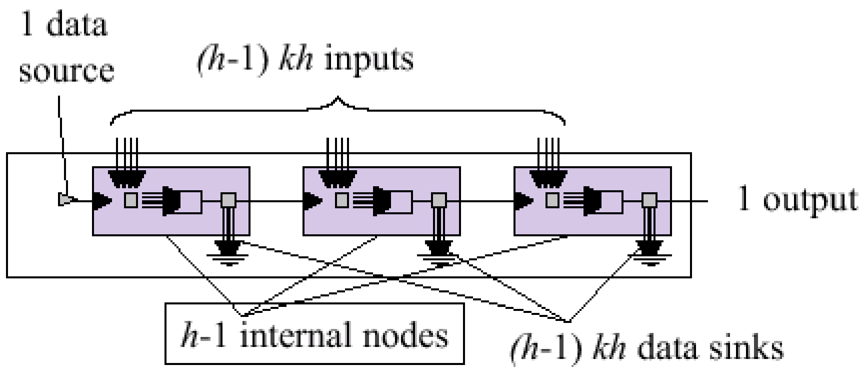
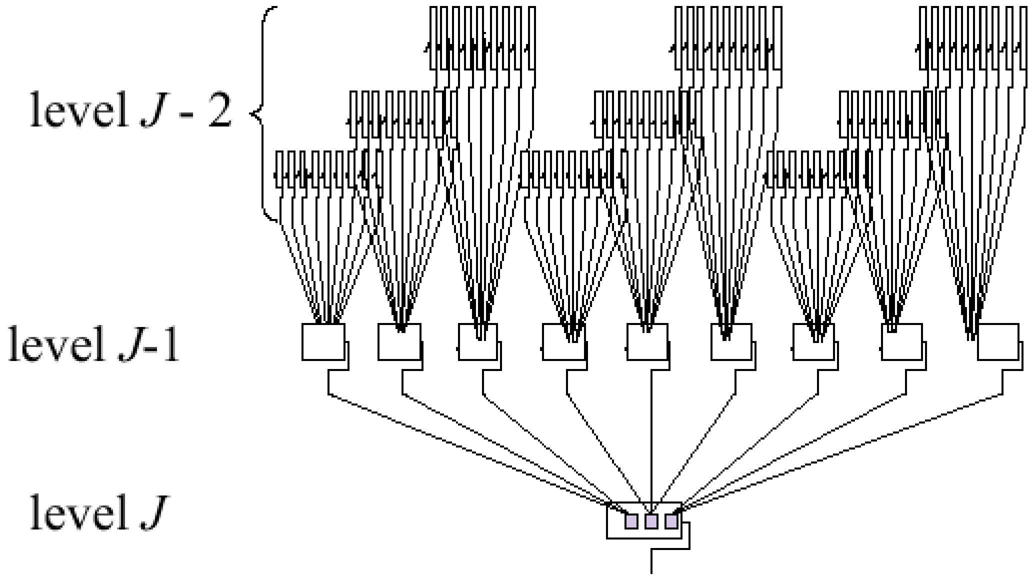


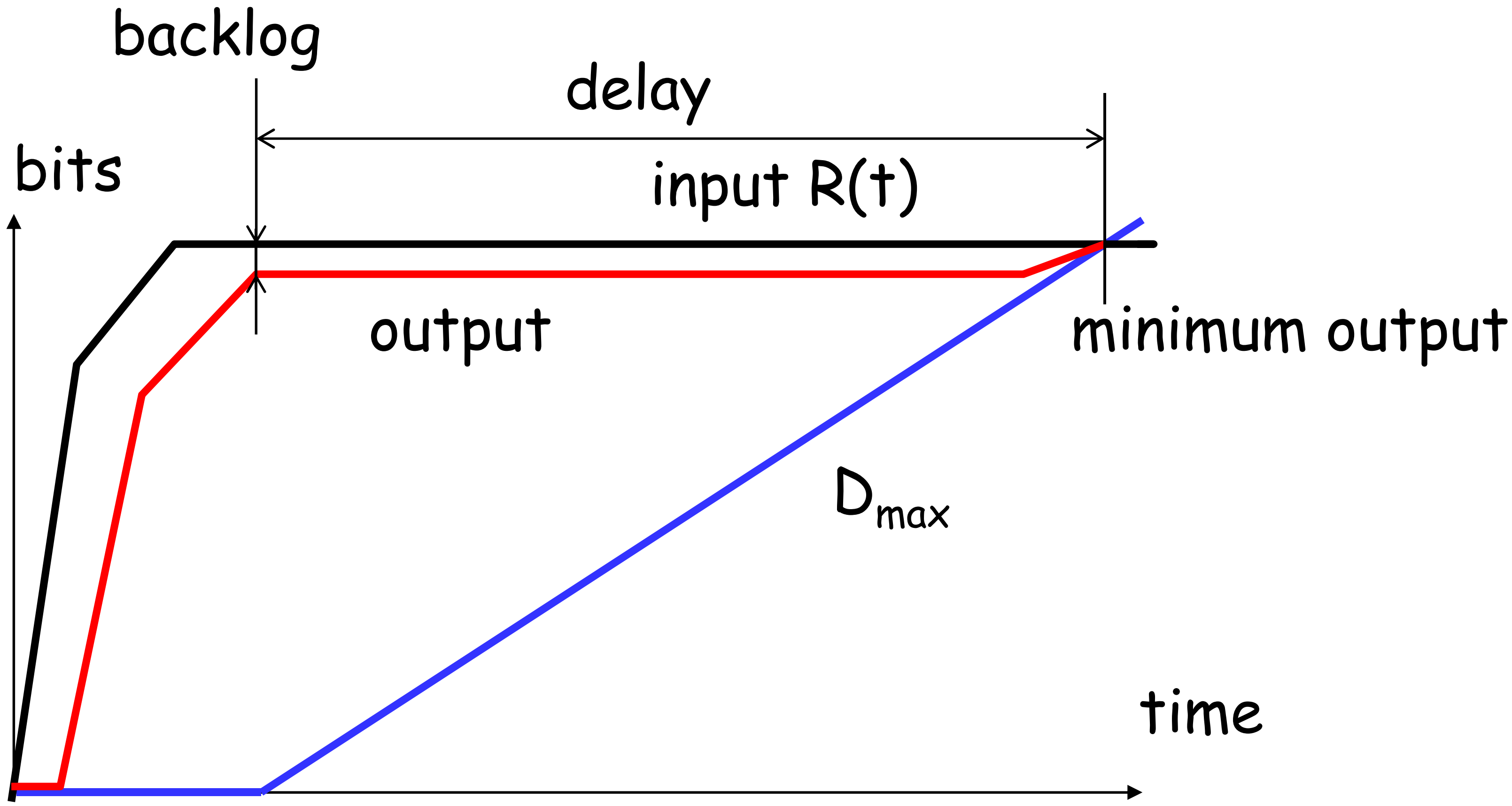
Figure 2.7: The bound D (in seconds) in Theorem 2.4.1 versus the utilization factor ν for $h = 10$, $e = 2\frac{1500B}{r_m}$, $L_{\max} = 1000$ b, $\sigma_i = 100$B and $\rho_i = 32$kb/s for all flows, $r_m = 149.760$Mb/s, and $C_m = +\infty$ (thin line) or $C_m = 2r_m$ (thick line).


buffer
multiplexer
demultiplexer

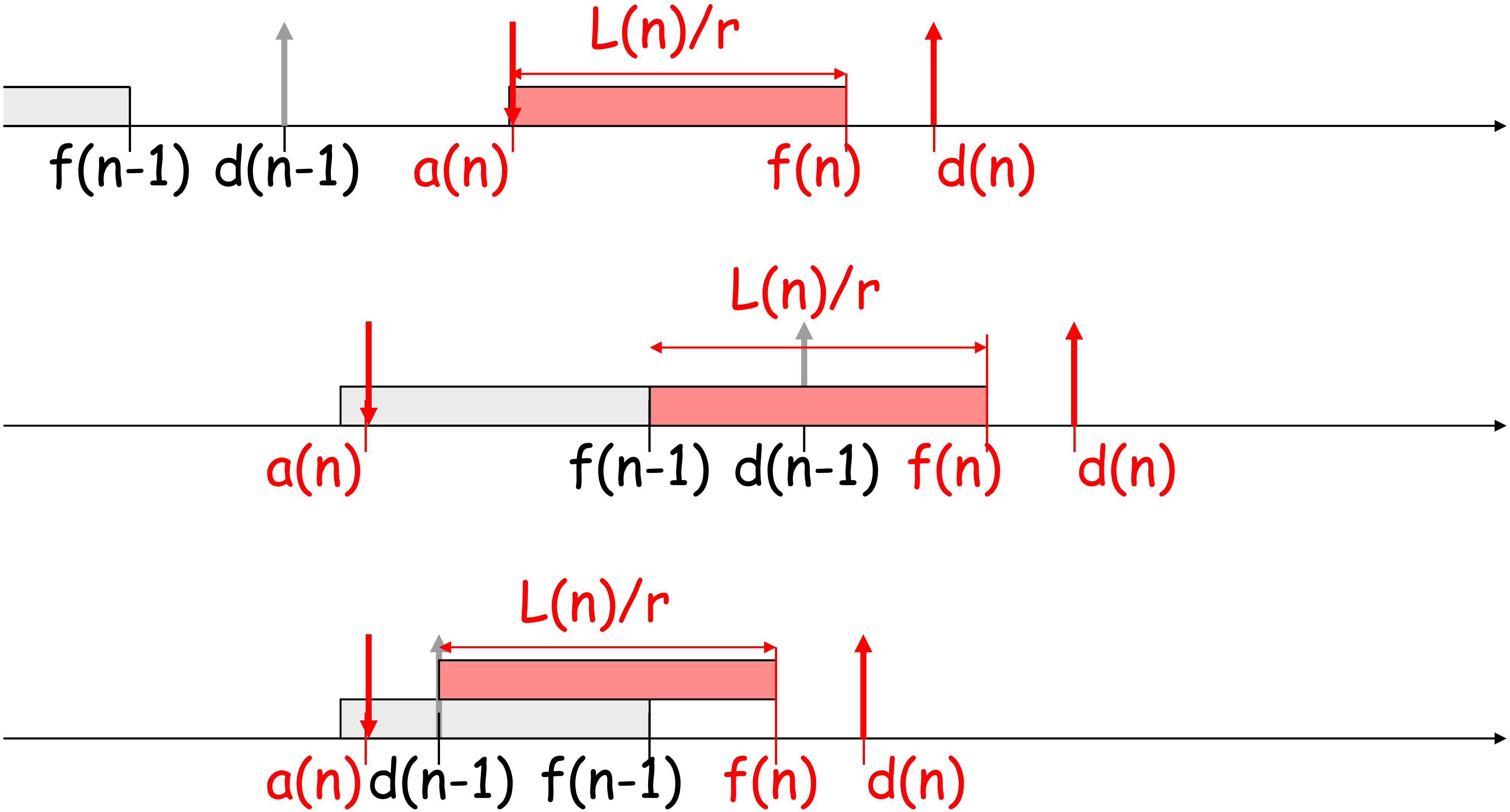

1 data
source
(h-1) kh inputs
1 output
h-1 internal nodes
(h-1) kh data sinks


level J - 2
level J-1
level J

GR and Service Curve do not Give Backlog from Delay Bound

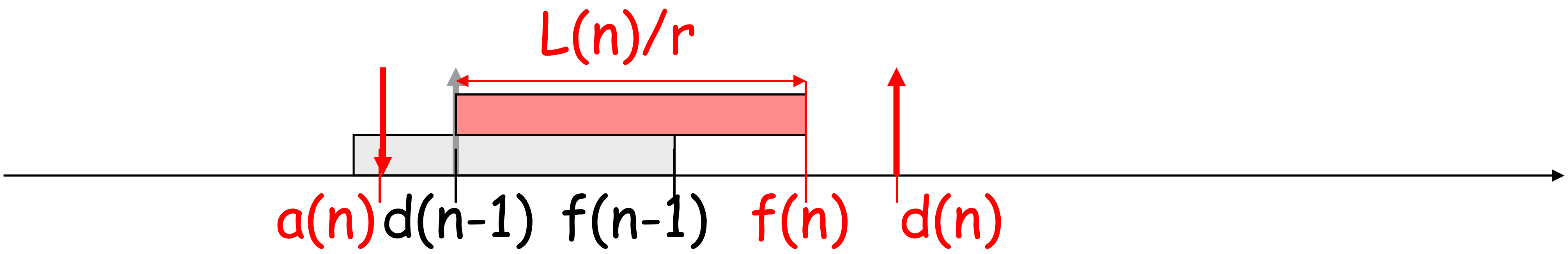


$$f(n) = \max\{a(n), \min[d(n-1), f(n-1)]\} + L(n)/r$$



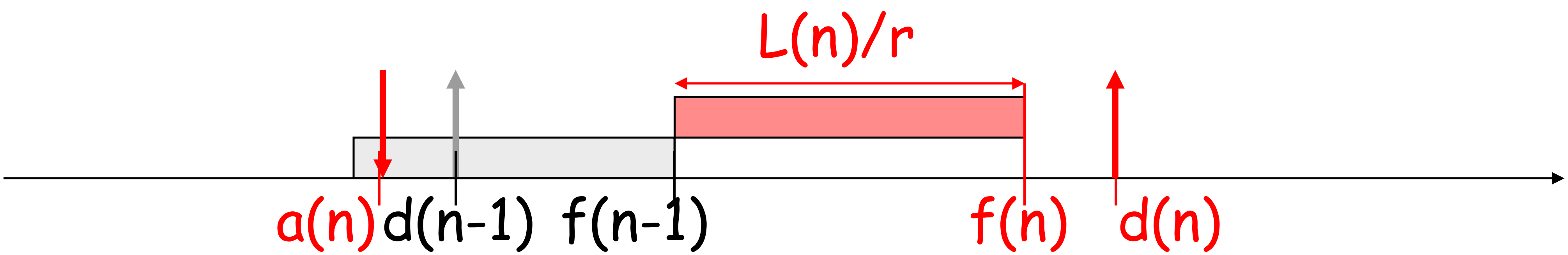
Packet Scale Rate Guarantee

$$f(n) = \max\{a(n), \min[d(n-1), f(n-1)]\}+ L(n)/r$$

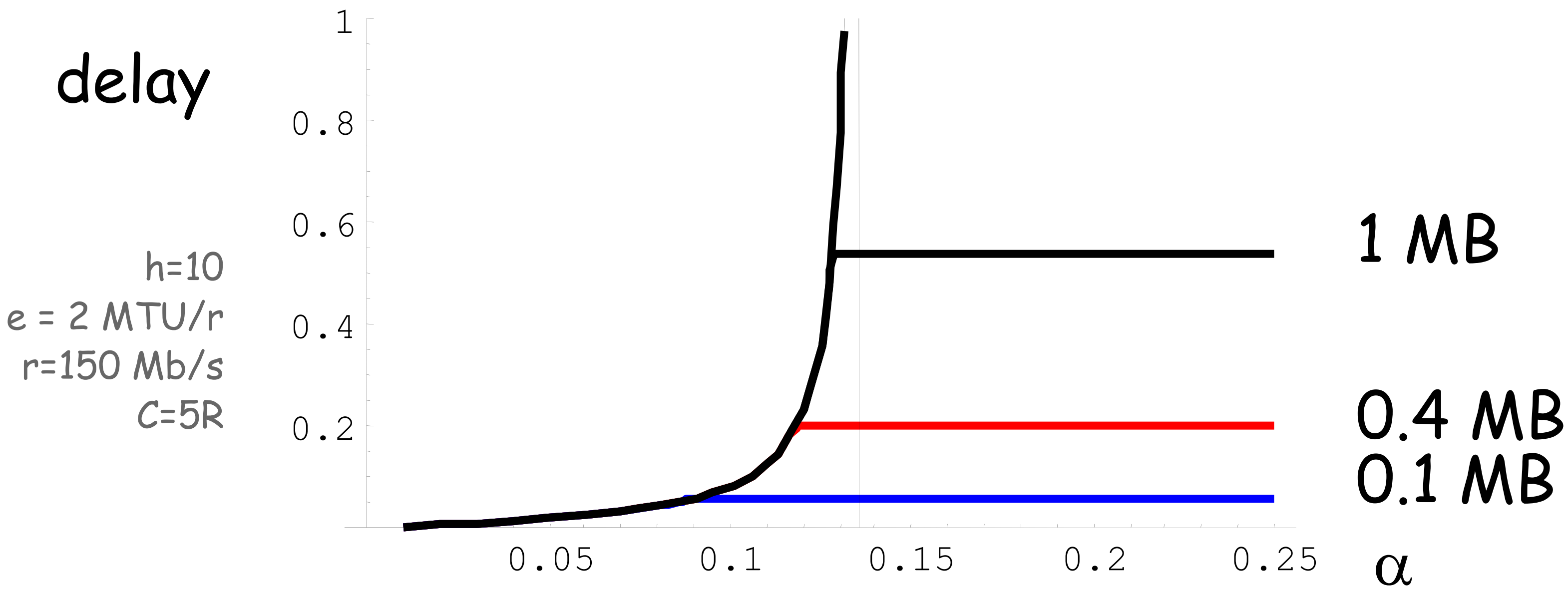


Guaranteed Rate

$$f(n) = \max\{a(n), f(n-1)]\}+ L(n)/r$$



Delay Bounds with PSRG versus with GR node model



Day 5

[14] I. Chlamtac, A. Faragó, H. Zhang, and A. Fumagalli. A deterministic approach to the end-to-end analysis of packet flows in connection oriented networks. *IEEE/ACM transactions on networking*, (6)4:422–431, 08 1998.

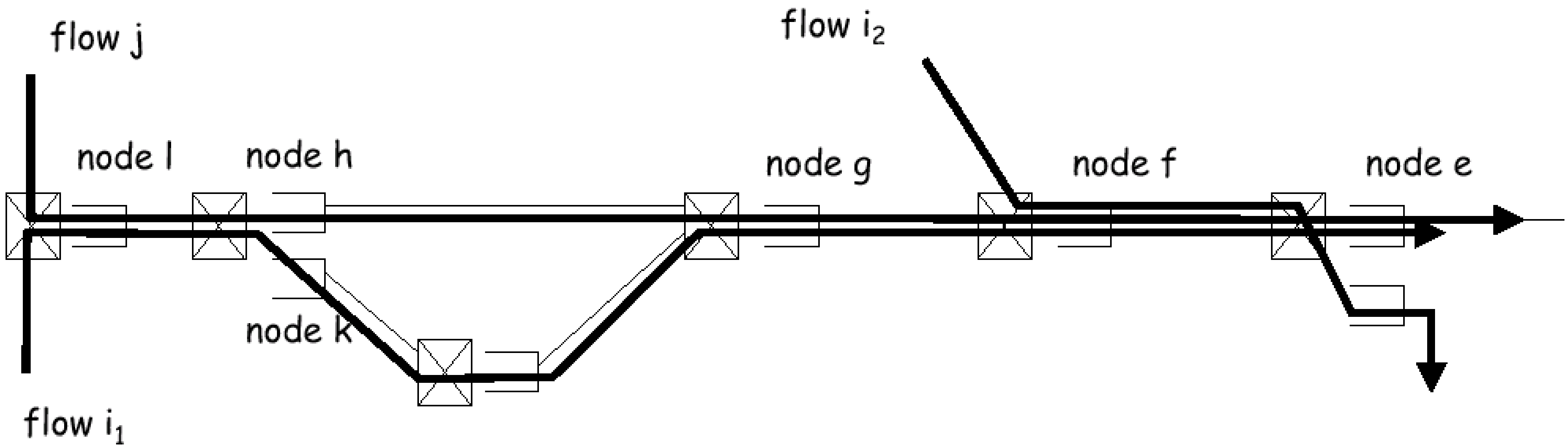


Figure 6.3: The network model and definition of an interference unit. Flows j and i_2 have an interference unit at node f. Flows j and i_1 have an interference unit at node l and one at node g.

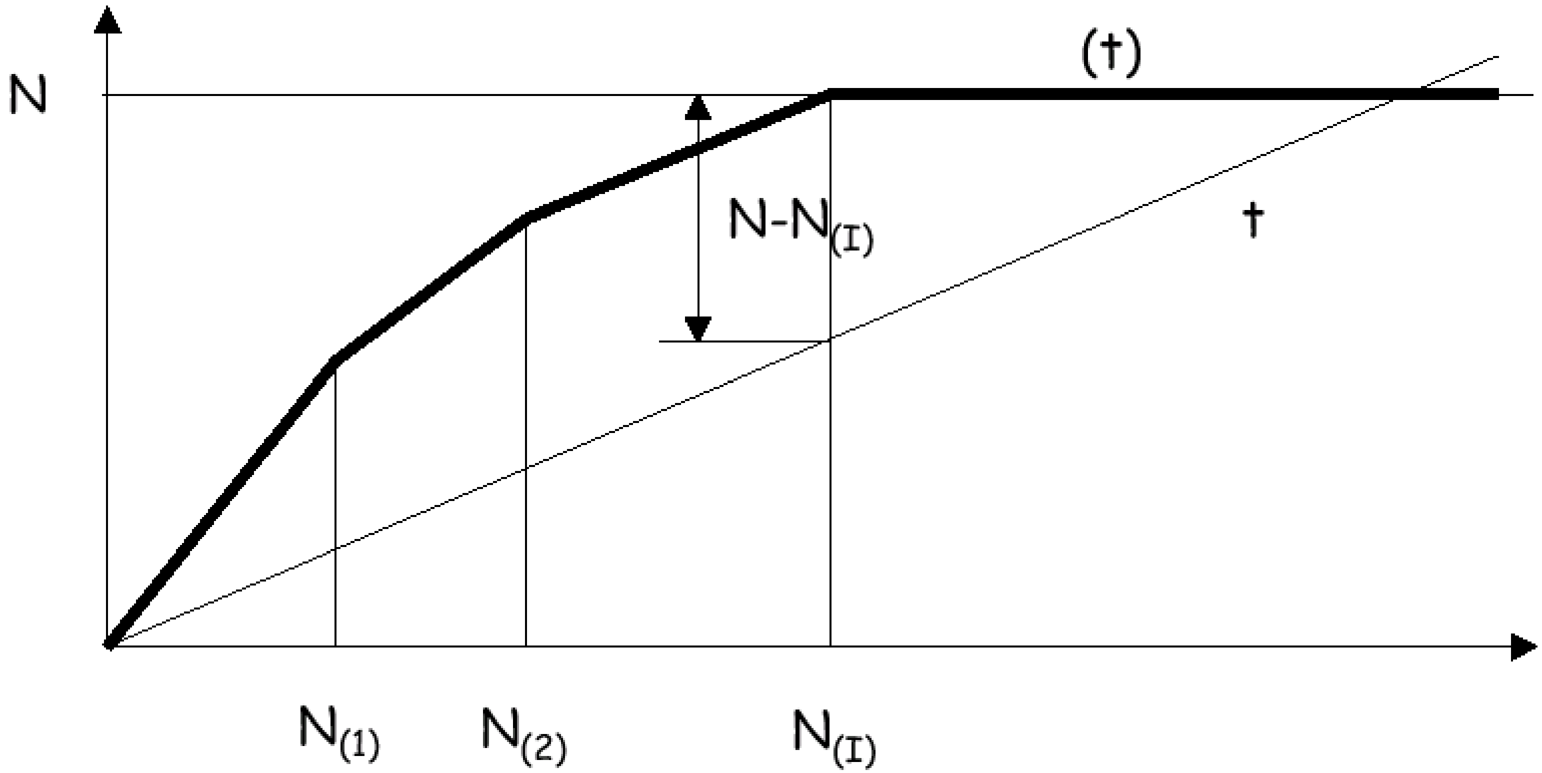
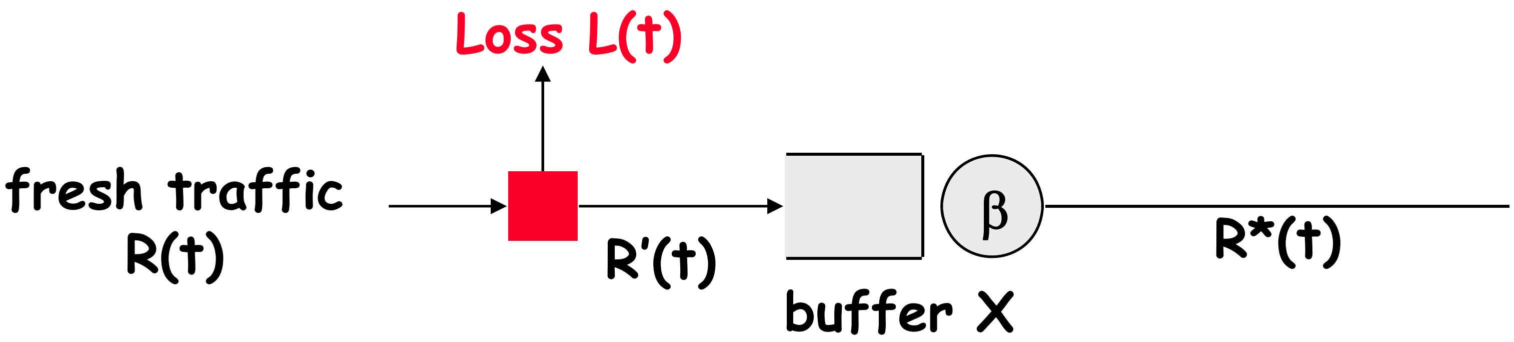


Figure 6.6: Derivation of a backlog bound.

Loss System



- node with service curve $\beta(t)$ and buffer X
- when buffer is full incoming data is discarded
- modelled by a virtual controller (*not* buffered)
- fluid model or fixed sized packets
- Pb: find loss ratio

Bound on Loss Ratio

- **Theorem** [Chuang and Cheng]: if R is α-smooth, then

$$L(t)/R(t) \leq 1 - r$$

with $r = \min(1, \inf_{t \geq 0} [\beta(t) + X] / \alpha(t))$

- best bound with these assumptions
- **proof**:
  - $R' \leq (X + \Pi(R')) \wedge h_R(R') \wedge \delta_0$ where Π is the transformation R' -> R, assumed isotone and usc (« physical assumptions ») R' is the maximum solution
  - define $x(t) = r\, R(t)$
  - x satisfies the system equation: $x \leq (X + x \otimes \beta) \wedge h_R(x) \leq (X + \Pi(x)) \wedge h_R(x)$
  - R' is the maximum solution => $x(t) \leq R'(t)$ for all t

Stochastic Bounds

- network calculus gives deterministic bounds on delay and loss
- combine with Hoeffding bounds [1963]:

  Assume
  - X_i are independent and $0 \leq X_i \leq 1$
  - $E(X_1 + \ldots + X_I) = s$ is known

  then for $s < x < I$

$$P(X_1 + \ldots + X_I > x) \leq \exp - \left(x \ln \frac{x}{s} + (I - x) \ln \frac{I - x}{I - s} \right)$$

Example: Bound on loss probability

- I independent, stationary sources with identical constraints σ_i served in a network element with *super-additive* service curve β
  [Chang, Vojnovic and L, Infocom 2002]

$$\boxed{P(Q > b) \le \inf_{\underline{s}} \left\{ \sum_k g(s_k, s_{k+1}) \right\}}$$

where $0 = s_0 < s_1 < \ldots < s_K = \tau$, $\tau = \inf\{t: \alpha(t) \le \beta(t)\}$

and for $\alpha(v) - \beta(u) > b$

$$g(u,v) = \exp\left(-I\left(\frac{\beta(u)+b}{\alpha(v)} \ln \frac{\beta(u)+b}{\rho v} + \frac{\alpha(v)-\beta(u)-b}{\alpha(v)} \ln \frac{\alpha(v)-\beta(u)-b}{\alpha(v)-\rho v} \right)\right)$$

else $g(u,v) = 0$

- Step 1: reduction to horizon τ

$$Q(0) = \sup_{s \le \tau} \{A(-s,0) - \beta(s)\}$$

- Step 2:

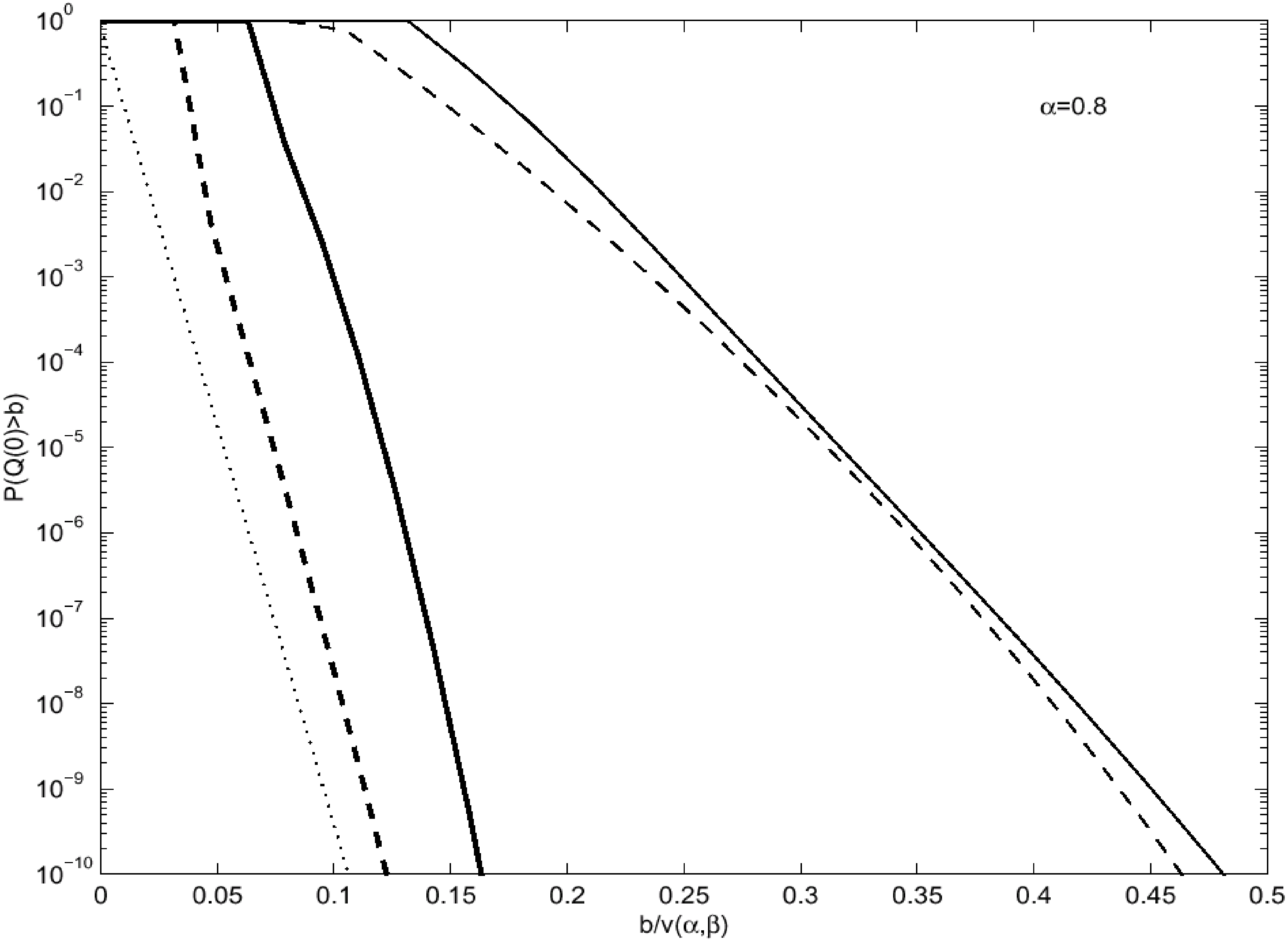
$$Q(0) = \max_k \{\sup_{s_k \le s \le s_{k+1}} A(-s,0) - \beta(s)\}$$
$$\le \max_k \{A(-s_{k+1},0) - \beta(s_k)\}$$

- Step 3 : Hoeffding to each term

$$A(-s_{k+1},0) - \beta(s_k) = \sum_i A_i(-s_{k+1},0) - \beta_i(s_k)$$

$$A_i(-s_{k+1},0) - \beta_i(s_k) \le \alpha_i(s_{k+1}) - \beta_i(s_k)$$

$$E\{A\ (-s_{k+1},0) - \beta(s_k)\} \le \rho s_{k+1} - \beta\ (s_k)$$



$$\alpha = 0.8$$

;. 5. A comparison with *Better than Poisson* approach. The input flows are homogeneously regulated. We fix the aggregate arrival curve to $\alpha(t) = \rho t + \sigma$ with $\rho = \alpha c$ and $\sigma = 500$ MTU; $c = 150$ Mbps, $e = 0$, MTU=1500 Bytes. The thick lines are for $I = 500$; the thin lines are for $I = 100$. Our bound (a) in (3) is shown as solid line; its homogeneous counterpart (Theorem 3 in [9]) as dashed line; and the asymptotic expansion for M/D/1 [7] as dotted line.