Network calculus
(system theory)

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- 2. the greedy shaper viewed as a min-plus system
- 3. min-plus operators and a theorem
- 4. the packetized shaper
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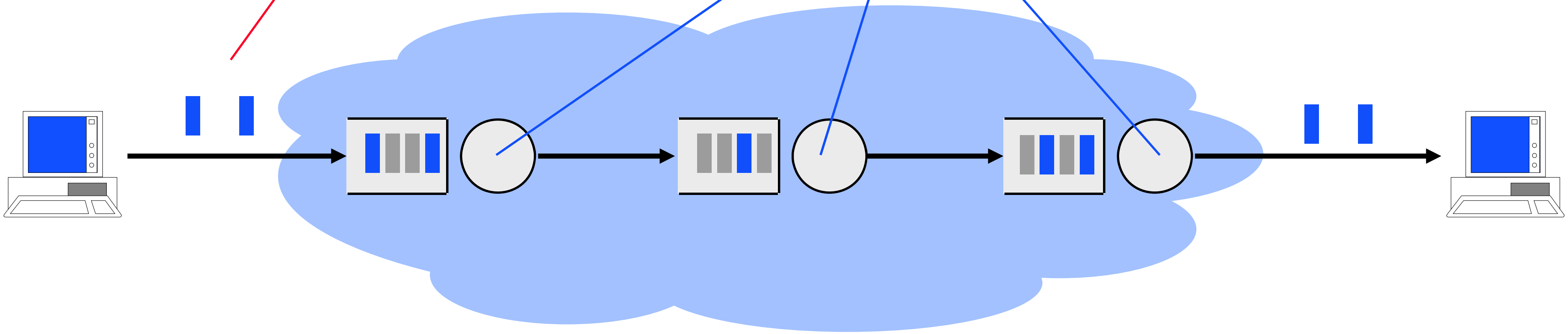
Part 1: Background Material
Arrival and Service Curves

- Internet integrated and differentiated services use the concepts of *arrival curve* and *service curves*

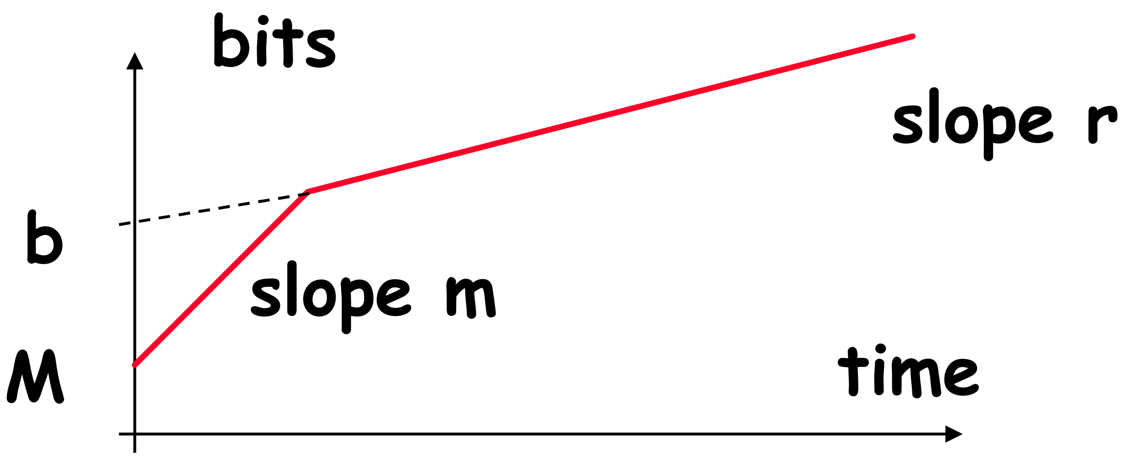


Arrival Curves

- Arrival curve α: $R(t) - R(s) \leq \alpha(t-s)$

Examples:

- leaky bucket $\alpha(u) = ru+b$
- standard arrival curve in the Internet
  $\alpha(u) = \min(pu+M, ru+b)$



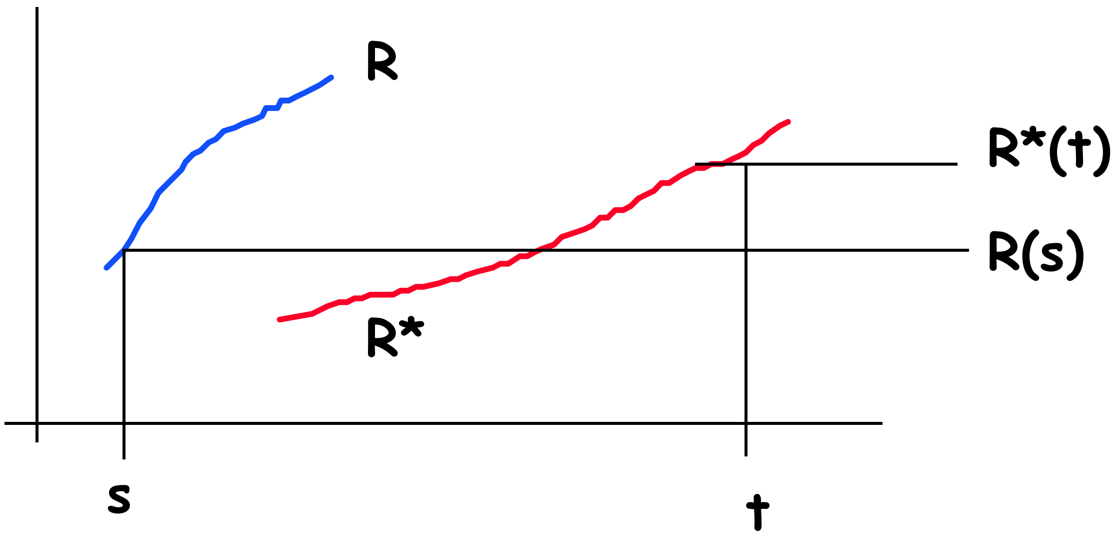
Arrival Curves can be assumed sub-additive

- **Theorem:** α can be replaced by a *sub-additive* function
- sub-additive: $\alpha(s+t) \leq \alpha(s) + \alpha(t)$
- concave => subadditive

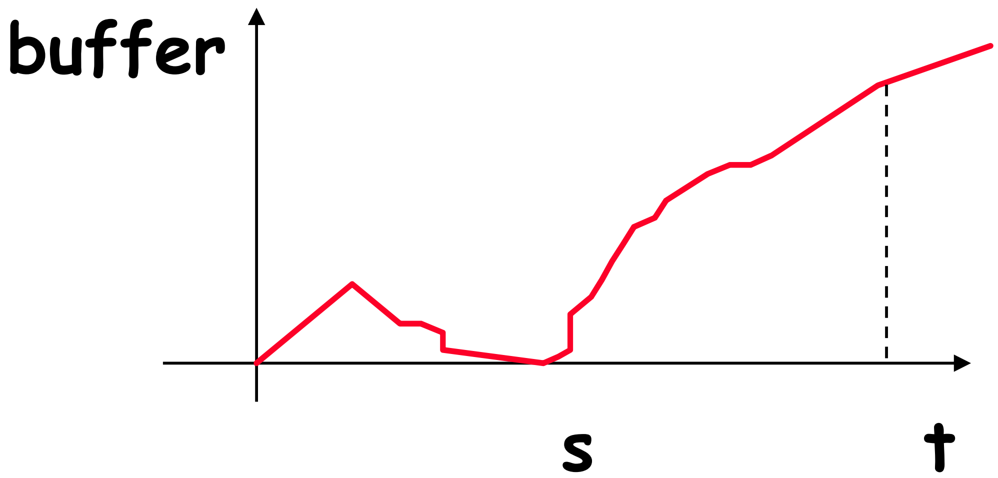
Service Curve

- System S offers a service curve β to a flow iff for all *t* there exists some *s* such that

$$R^*(t) - R(s) \geq \beta(t - s)$$



The constant rate server has service curve $\beta(t)=ct$

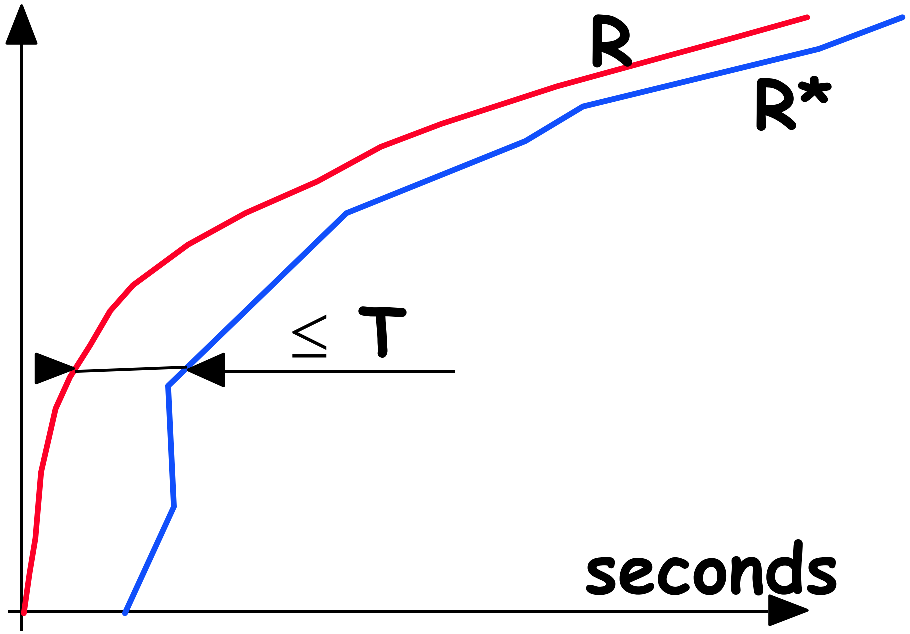


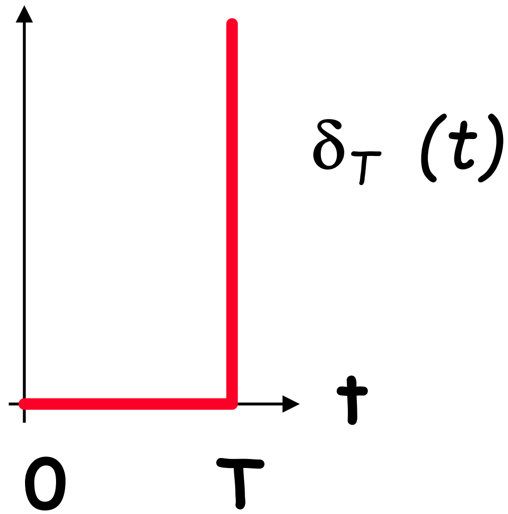
Proof: take s = beginning of busy period:

$$R^*(t) - R^*(s) = c\ (t-s)$$

$$R^*(t) - R(s) = c\ (t-s)$$

The guaranteed-delay node has service curve δ_T

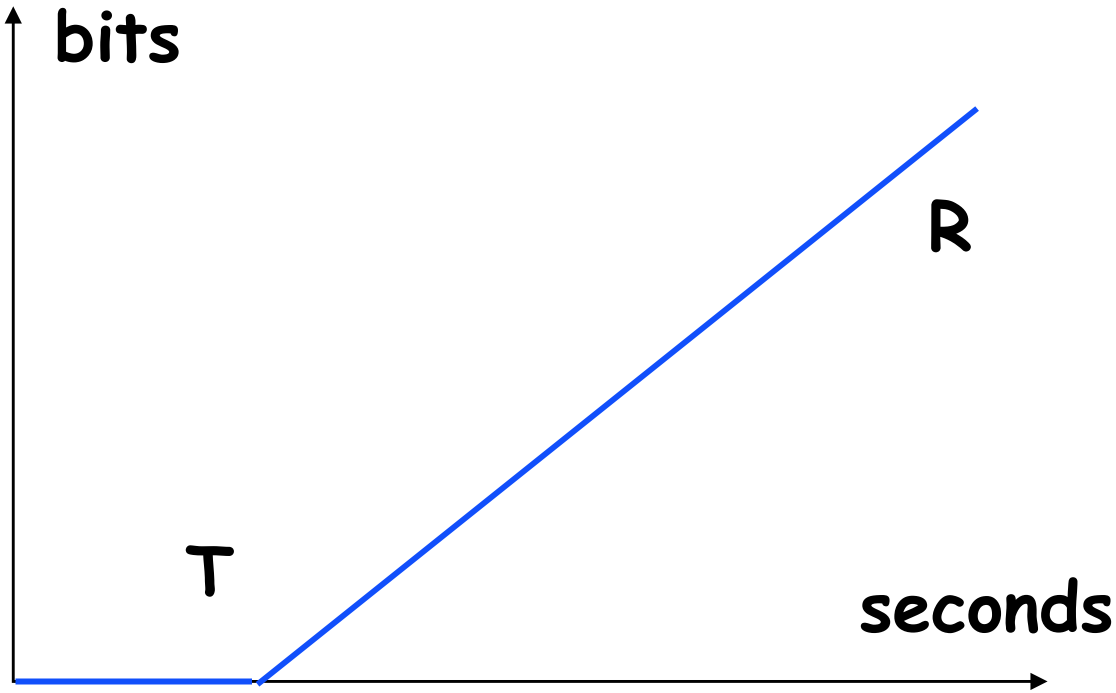




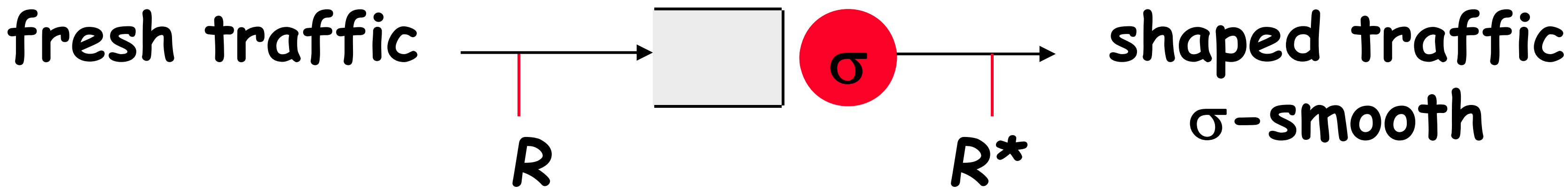
Function δ_T

The standard model for an Internet router

- rate-latency service curve

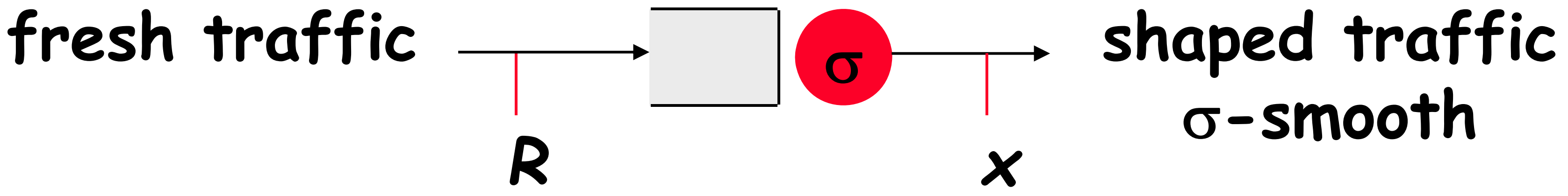


Part 2: The Greedy Shaper viewed as a Min-Plus System



- shaper: forces output to be constrained by σ
- greedy shaper stores data in a buffer only if needed
- examples:
  - constant bit rate link ($\sigma(t)=ct$)
  - ATM shaper; fluid leaky bucket controller
- Pb: find input/output relation

A Min-Plus Model for Shapers



- Shaper Equations:
  - (1) $x \le x \otimes \sigma$
  - (2) $x \le R$

(1) $x \le x \otimes \sigma$

(2) $x \le R$

Solving the Min-Plus Model

- Theorem: There is a maximum solution; it is equal to $R \otimes \sigma$
- Proof:

(1) find a solution: fixed point

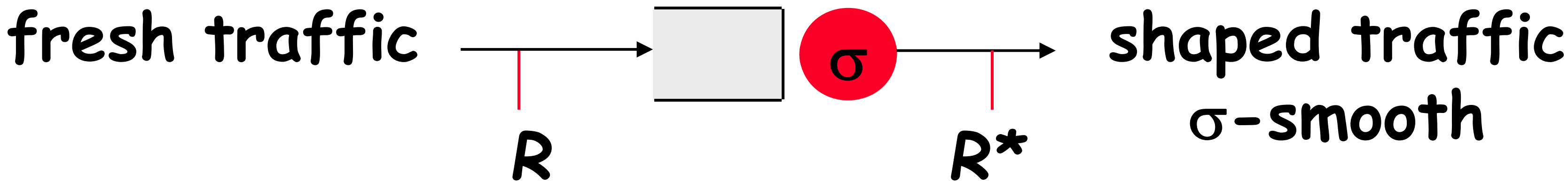
$x^0 = R$; $x^i = x^{i-1} \otimes \sigma$ and $x^* = \inf \{x^0, x^1, ..., x^i, ...\}$

here: $\sigma \otimes \sigma = \sigma$ and thus $x^i = R \otimes \sigma = x^*$

(2) if x is a solution, then

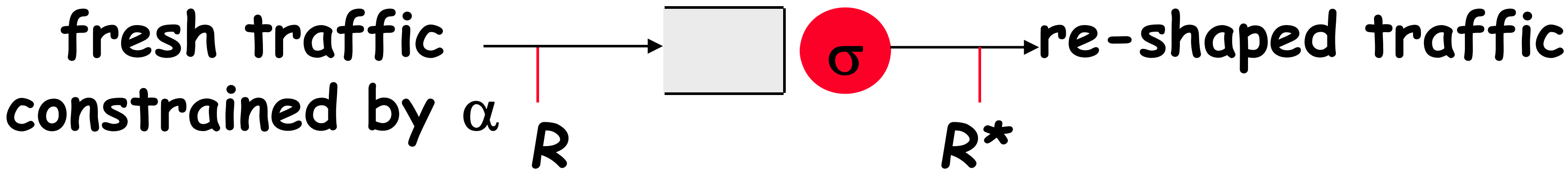
$x \le R$ thus $x \otimes \sigma \le R \otimes \sigma$

I/O of Greedy Shaper



- for any shaper, output $\leq R \otimes \sigma$
- R is wide-sense increasing, thus $R \otimes \sigma$ also
- thus: the greedy shaper output is $R^* = R \otimes \sigma$

A consequence: Greedy Shaper Keeps Arrival Constraints



- The output of the shaper is still constrained by α
- **Proof**

$$R^* = R \otimes \sigma \leq (R \otimes \alpha) \otimes \sigma = (R \otimes \sigma) \otimes \alpha = R^* \otimes \alpha$$

Part 3: Min-Plus Operators and a theorem

- G = set of functions R -> R$^+$ that are wide-sense increasing
- works also if time is discrete: N -> R$^+$
- we consider operators $\Pi : G \to G$
- Π is isotone if $x(t) \leq y(t) \Rightarrow \Pi(x)(t) \leq \Pi(y)(t)$
- Π is upper-semi continuous iff $\inf_i(\Pi(x_i)) = \Pi(\inf_i(x_i))$ for $\downarrow$ sequences x_i

Min-Plus Linear Operators

- Π is min-plus linear if
  - for any constant K, $\Pi(x + K) = \Pi(x) + K$
    $$\Pi(x \wedge y) = \Pi(x) \wedge \Pi(y)$$
  - Π is upper-semi continuous.
- Representation Theorem: Π is min-plus linear <=> there is some H: R x R -> $\underline{R}^+$ such that $\Pi(x)(t)=\inf_s[H(t,s)+x(s)]$
- min-plus linear => isotone

Other Properties of Operators

- Π is *time invariant* if for some T
  $y(t) = \Pi(x)(t)$ and $x'(t) = x(t+T)$ => $\Pi(x')(t) = y(t+T)$
- Π is *causal* if $\Pi(x)(t)$ depends only on $x(s)$, $0 \le s \le t$

Two linear operators

- Convolution by a fixed function:
  C_{σ}: $x \rightarrow x \otimes \sigma$
  - C_{σ} is linear, time invariant, not causal
  - $C_{\sigma} \circ C_{\sigma'} = C_{\sigma \otimes \sigma'}$
- Idempotent operator h_M $x \rightarrow h_M(x)$

  with $h_M(t) = \inf_{s \leq t} \{ M(t) - M(s) + x(s) \}$

  - is idempotent: $h_M \circ h_M = h_M$
  - linear, causal, not time invariant

The Packetizer

- Define *function* P^L by
  $P^L(x) = L(n) \Leftrightarrow L(n) \leq x < L(n+1)$ [Chang 99]
- call P_L the operator: $P_L(R)(t) = P^L(R(t))$ accumulates bits until entire packets can be delivered
- P_L is idempotent, not linear, but is isotone and upper-semi continuous

A Min-Plus Theorem

- Implicitely contained in Baccelli et al, "Synchronization and Linearity", Baccelli et al.
- **Theorem**: Assume that Π is *isotone* and *upper-semi-continuous*. The problem

  $$x(t) \leq b(t) \wedge \Pi(x)(t)$$

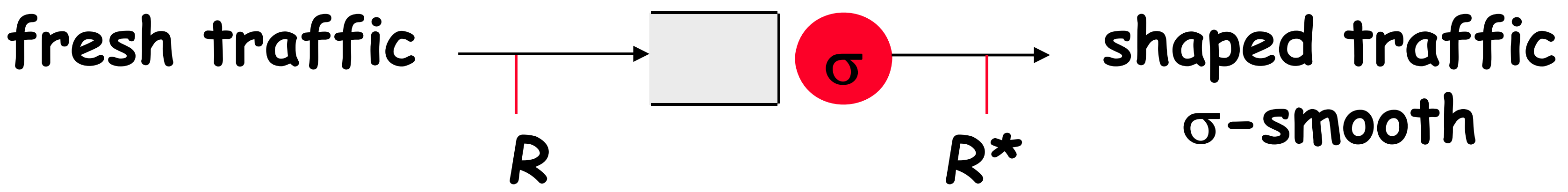
  has one maximum solution in G, given by

  $$x^*(t) = \underline{\Pi}(b)(t)$$
- (Definition of closure)

  $$\underline{\Pi}(x) = \inf \{x, \Pi(x), \Pi o \Pi(x), \Pi o \Pi o \Pi(x), \dots\}$$
- in other words:

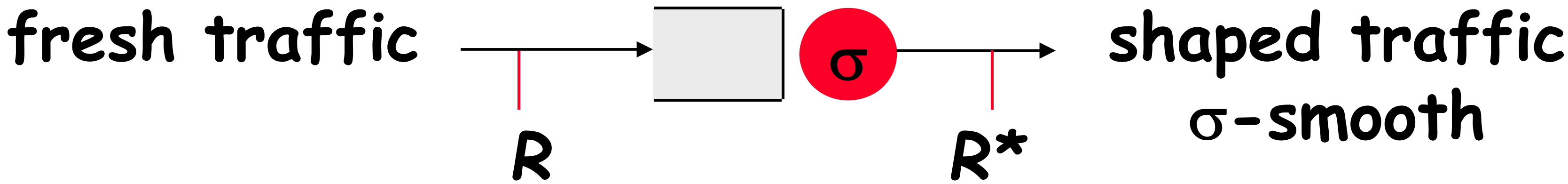
  $x^0 = b$; $x^i = \Pi(x^{i-1})$ and $x^* = \inf \{x^0, x^1, \dots, x^i, \dots\}$

Part 4: packetized shaper



- same as previous, but releases only entire packets
- example : leaky bucket controller
- Pb: find input/output relation of packetized greedy shaper

Model for packetized shapers



- Define $L(i) = l_1 + l_2 + \ldots + l_i$
- The output satisfies:
  - (1) $R^* \leq R^* \otimes \sigma$
  - (2) $R^* \leq R$
  - (3) R^* is L-packetized

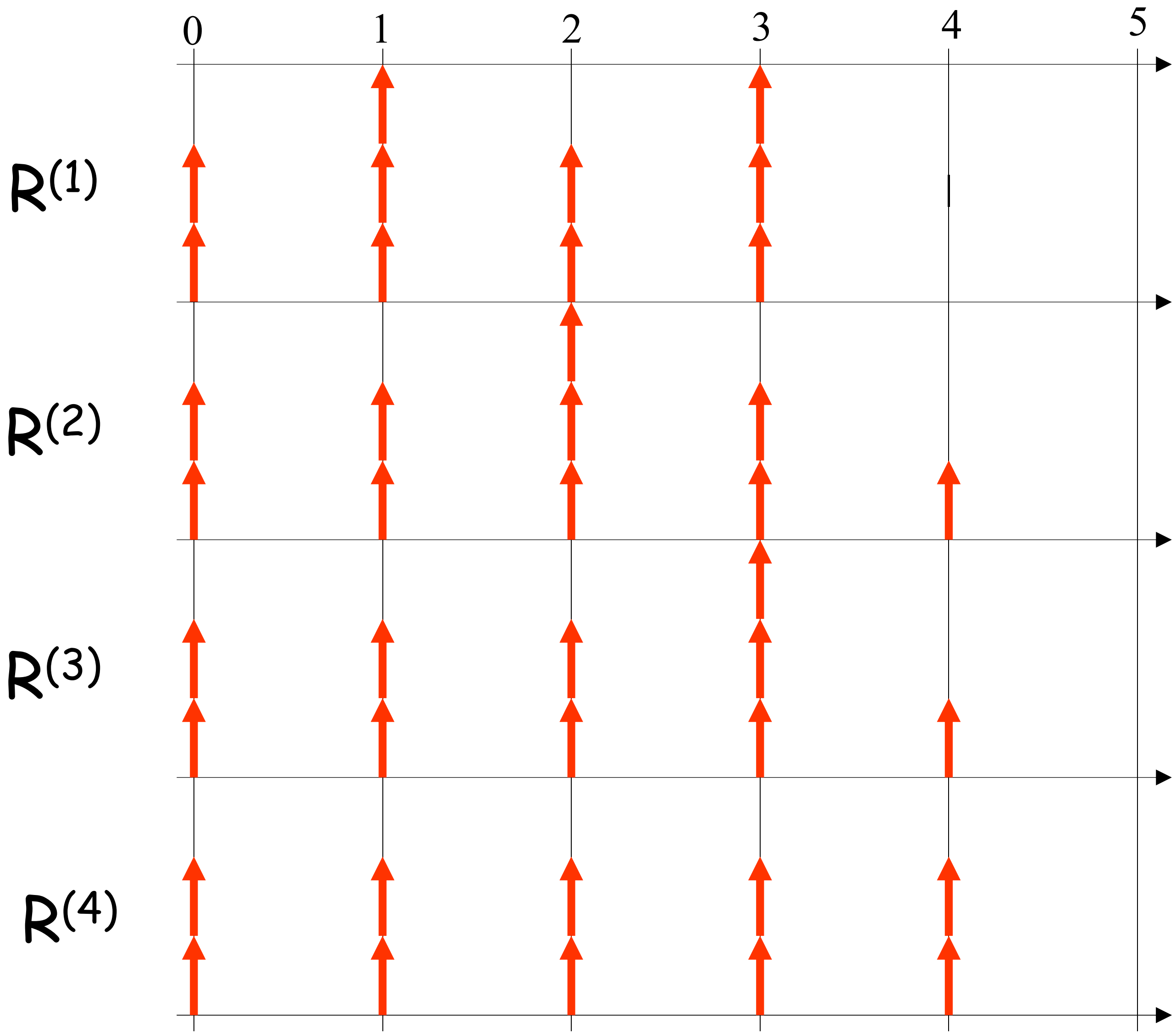
Modelling packetized greedy shapers

- system equation : $R^* \le P_L(R^*) \wedge C_\sigma(R^*) \wedge R$
- maximum solution: $R^* = \underline{P_L \wedge C_\sigma}(R)$
- th 4.3.3: closure $((P \wedge Id) o (Q \wedge Id))$= closure$(P \wedge Q)$
  thus closure$(P_L \wedge C_\sigma)$=closure$(P_L \circ C_\sigma)$
- after some algebra:
  $R^* = \inf \{R^{(1)}, R^{(2)}, R^{(3)}, \ldots\}$
  with $R^{(i)} = P_L \circ C_\sigma \circ \ldots \circ P_L \circ C_\sigma (R)$

  i.e. $R^{(0)} = R$, $R^{(i)} = P^L(R^{(i-1)} \otimes \sigma)$

Numerical Example for $R^* = \underline{P_L \triangle C_\sigma}(R)$

- $\sigma(t) = 25 \lceil t/T \rceil$ for $t > 0$, else 0
- σ – smooth <=> at most 25 data unit per time unit
- $R(t)$= a burst of 10 packets of size 10 at time 0
- $R^{(i)} = P^L(R^{(i-1)} \otimes \sigma)$



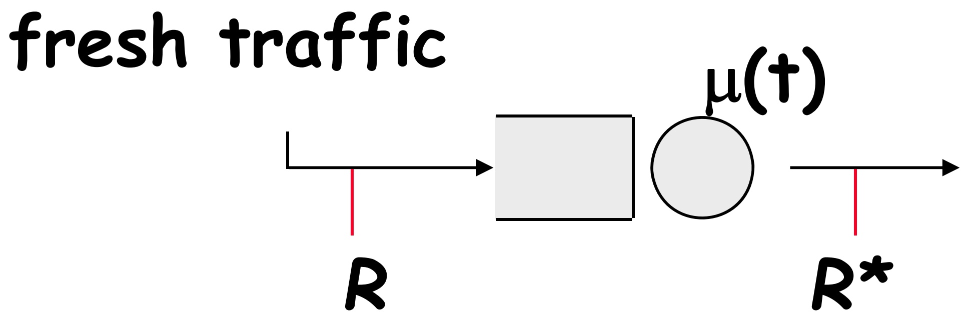
Special Case

- **Theorem** (LeBoudec, Sigmetrics 2001)
  If $\sigma = \sigma_0 + l$ with $l \geq l_{max}$ then

  $$P_L \circ C_\sigma \circ \ldots \circ P_L \circ C_\sigma = P_L \circ C_\sigma \circ P_L$$

  and thus $R^* = P^L(R \otimes \sigma)$
- Applications: if σ is concave and $\sigma(0+) \geq l_{max}$ then the packetized shaper can be realized as the concatenation : shaper + packetizer
- leaky bucket controllers based on bucket replenishment are functionally equivalent to leaky bucket based on virtual finish times

Part 5: Other Examples
Ex3: Variable Capacity Node



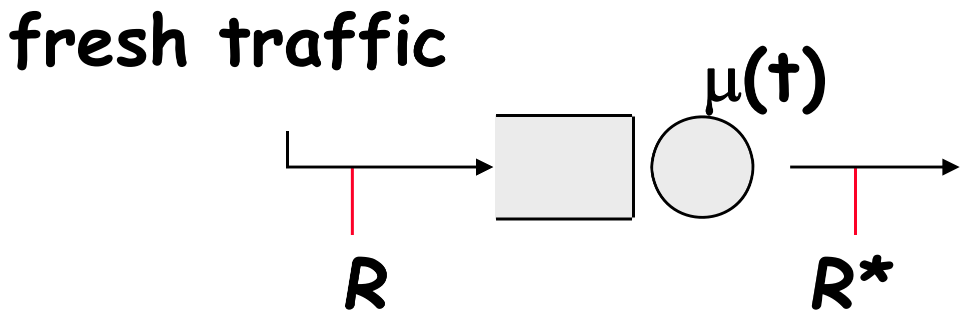
- node has a time varying capacity $\mu(t)$
  Define $M(t) = \int_0^t \mu(s)\, ds$.
- the output satisfies

  $R^* \leq R$

  $R^*(t) - R^*(s) \leq M(t) - M(s)$ for all $s \leq t$

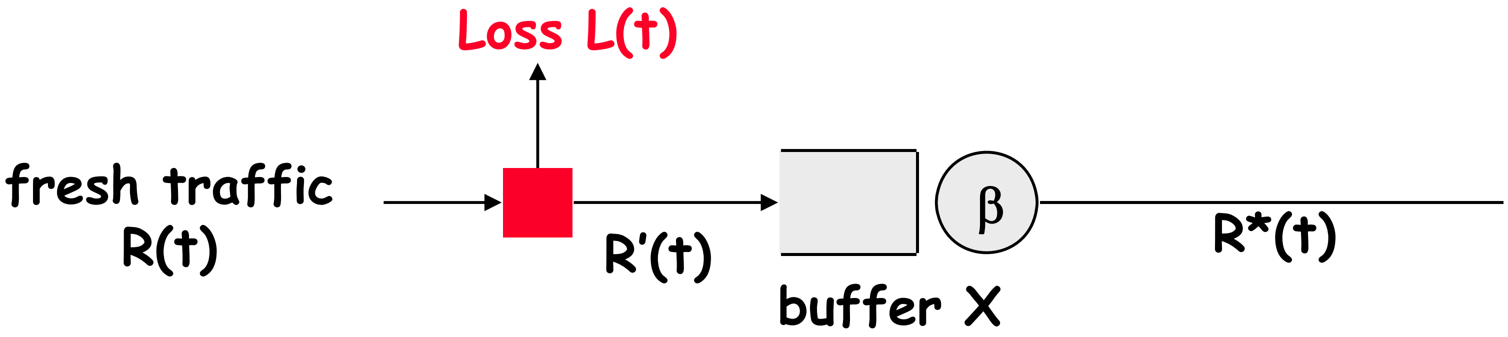
and is "as large as possible"

Variable Capacity Node



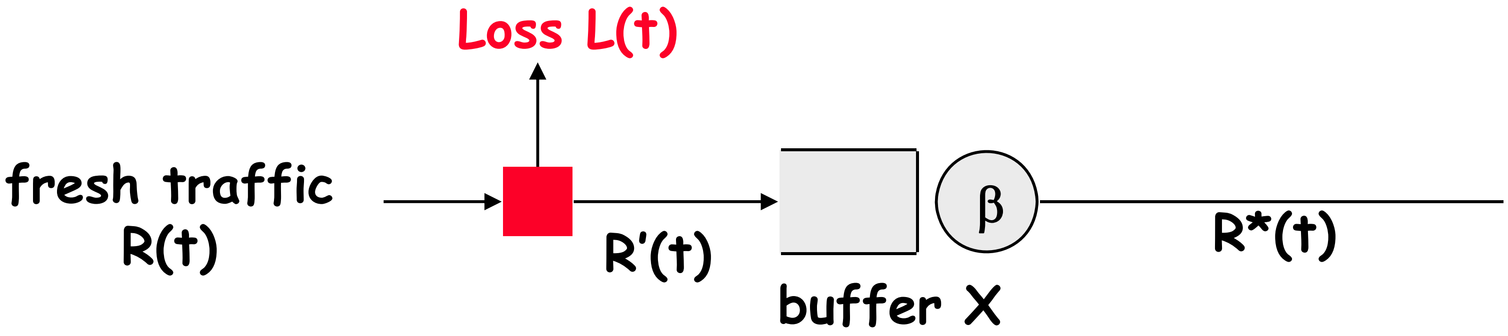
- $R^* \leq R$
  $R^*(t) - R^*(s) \leq M(t) - M(s)$ for all $s \leq t$
- $R^* \leq R \wedge h_M(R^*)$
- thus there is a maximum solution in G, and $R^* = \underline{h}_M(R)$
- now h_R is idempotent thus $\underline{h}_M = h_M$
- finally: $R^*(t) = \inf_{s \leq t} \{ M(t) - M(s) + R(s) \}$

Ex 4: Loss System



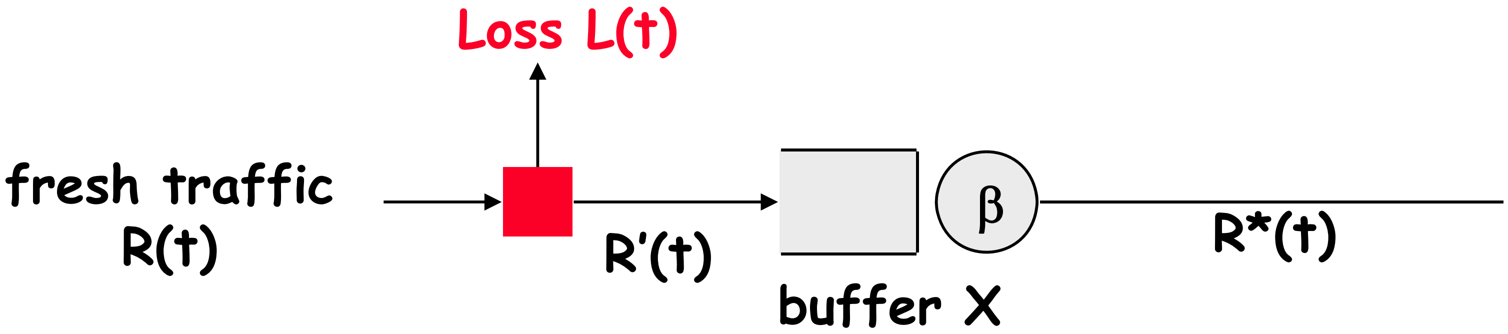
- node with service curve $\beta(t)$ and buffer X
- when buffer is full incoming data is discarded
- modelled by a virtual controller (*not* buffered)
- fluid model or fixed sized packets
- Pb: find loss ratio

Model for Loss System



- R'(t) satisfies
  $R' \le (X + \Pi(R')) \wedge h_R(R') \wedge \delta_0$ where Π is the transformation R' -> R*
- assume Π isotone and usc (« physical assumptions »); thus $R' = \underline{(X + \Pi) \wedge h_R}(\delta_0)$
- we don't know Π but $\Pi \ge C_\beta$
- <u>theorem</u>: $\Pi \ge \Pi' \Rightarrow \underline{\Pi} \ge \underline{\Pi'}$
- thus $R' \ge \underline{(X + C_\beta) \wedge h_R}(\delta_0)$

Representation of Loss



- we have shown: $R' \geq \underline{(X + \mathcal{C}_{\beta}) \wedge h}_{\underline{R}}(\delta_0)$
- compute the closure, obtain R', thus the loss process L=R-R'

$$L(t) \leq \sup_{k \geq 0;\, 0 \leq s_{2k} \leq .. \leq s_2 \leq s_1 \leq t} \left\{ \sum_{i=1}^{k} \left[R(s_{2i-1}) - R(s_{2i}) - \beta(s_{2i-1} - s_{2i}) \right] - kX \right\}$$

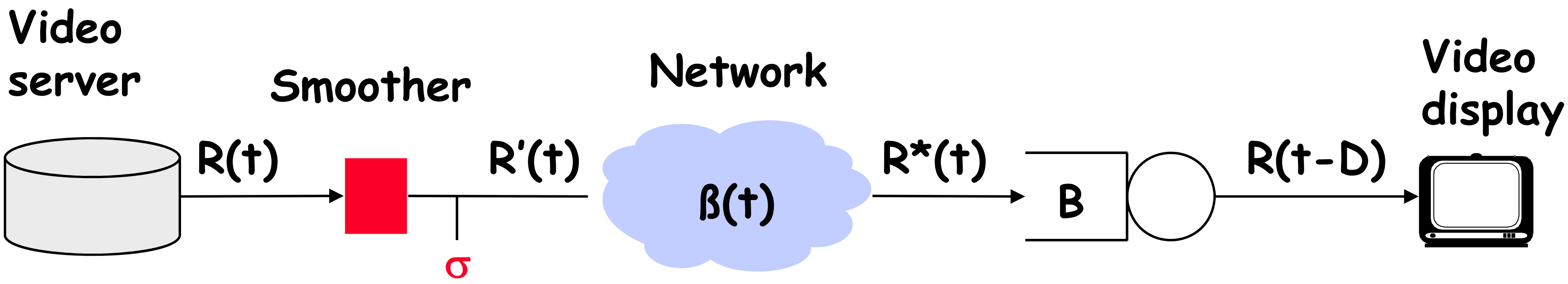
Bound on Loss Ratio

- **Theorem**: if R is α-smooth, then

  $$L(t)/R(t) \le 1 - r$$

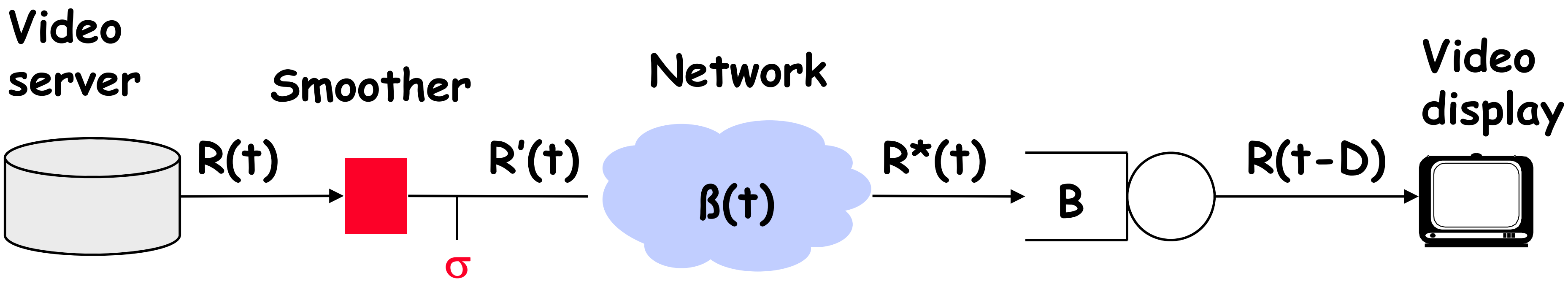
  with $r = \min(1, \inf_{t \ge 0} [\beta(t) + X] / \alpha(t))$
- best bound with these assumptions
- **proof**:
  - define $x(t) = r\, R(t)$
  - x satisfies the system equation:
    $x \le (X + x \otimes \beta) \wedge h_R(x) \le (X + \Pi(x)) \wedge h_R(x)$
  - R' is the maximum solution
    => $x(t) \le R'(t)$ for all t

Ex 5: Optimal smoothing



- Network offers a service curve β to flow *R'(t)*,
- Smoother delivers a flow *R'(t)* conforming to an arrival curve σ.
- Video stream is stored in the client buffer *B* read after a playback delay *D*.
- Pb: which smoothing strategy minimizes *D* and *B* ?

System Equations



- (1) R' is σ-smooth
- (2) $(R' \otimes \beta)(t) \geq R(t-D)$
- $R'(t) = 0$ for $t \leq 0$

- Define min-plus deconvolution
  $(a \oslash b)(t) = \sup_{s \geq 0} [a(s+t)-b(s)]$
- $x \leq y \otimes \beta \iff x \oslash \beta \leq y$

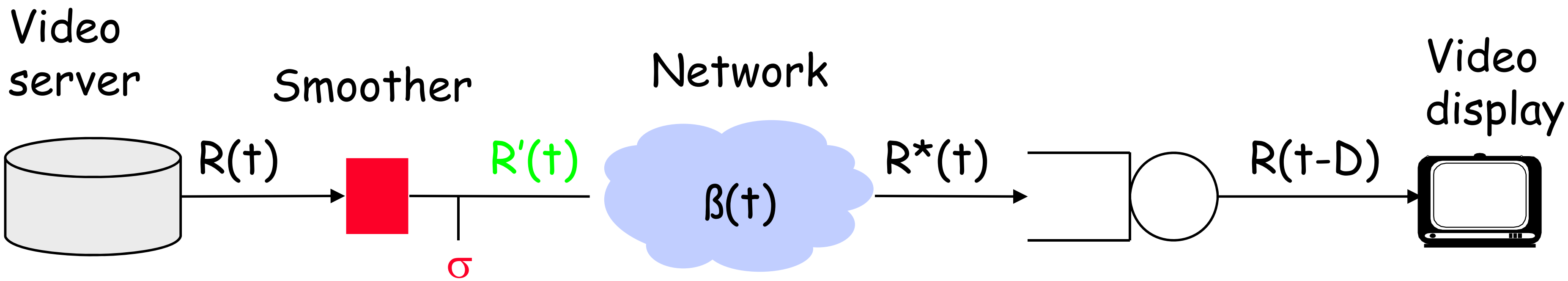
Max-Plus operators

- replace $\leq$ by $\geq$, min-plus becomes max-plus
- Deconvolution with a fixed function

  $x \to x \oslash a$

  is max-plus linear

A max-plus model for Example 5



- R' satisfies:

  (1) $R' \geq R' \oslash \sigma$

  (2) $R' \geq (R \oslash \beta)(t-D)$

- a max-plus system, , with *minimum* solution

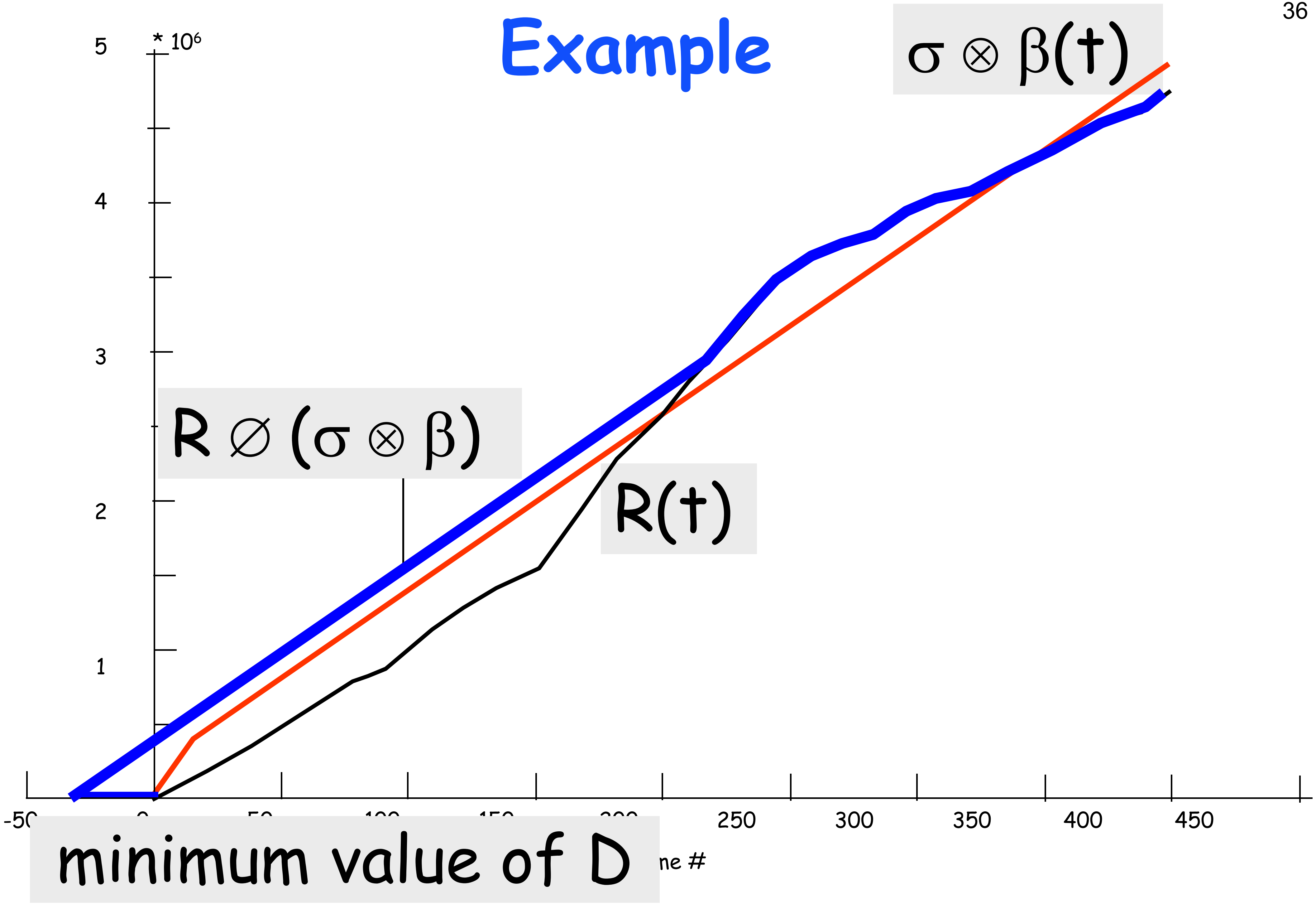
  $x^* = \inf \{x^0, x^1, \ldots, x^i, \ldots\}$

  $x^0(t) = (R \oslash \beta)(t-D)$

  $x^i = x^{i-1} \oslash \sigma$

- thus $R' = (R \oslash \beta) \oslash \sigma\ (t-D) = R \oslash (\beta \otimes \sigma)\ (t-D)$

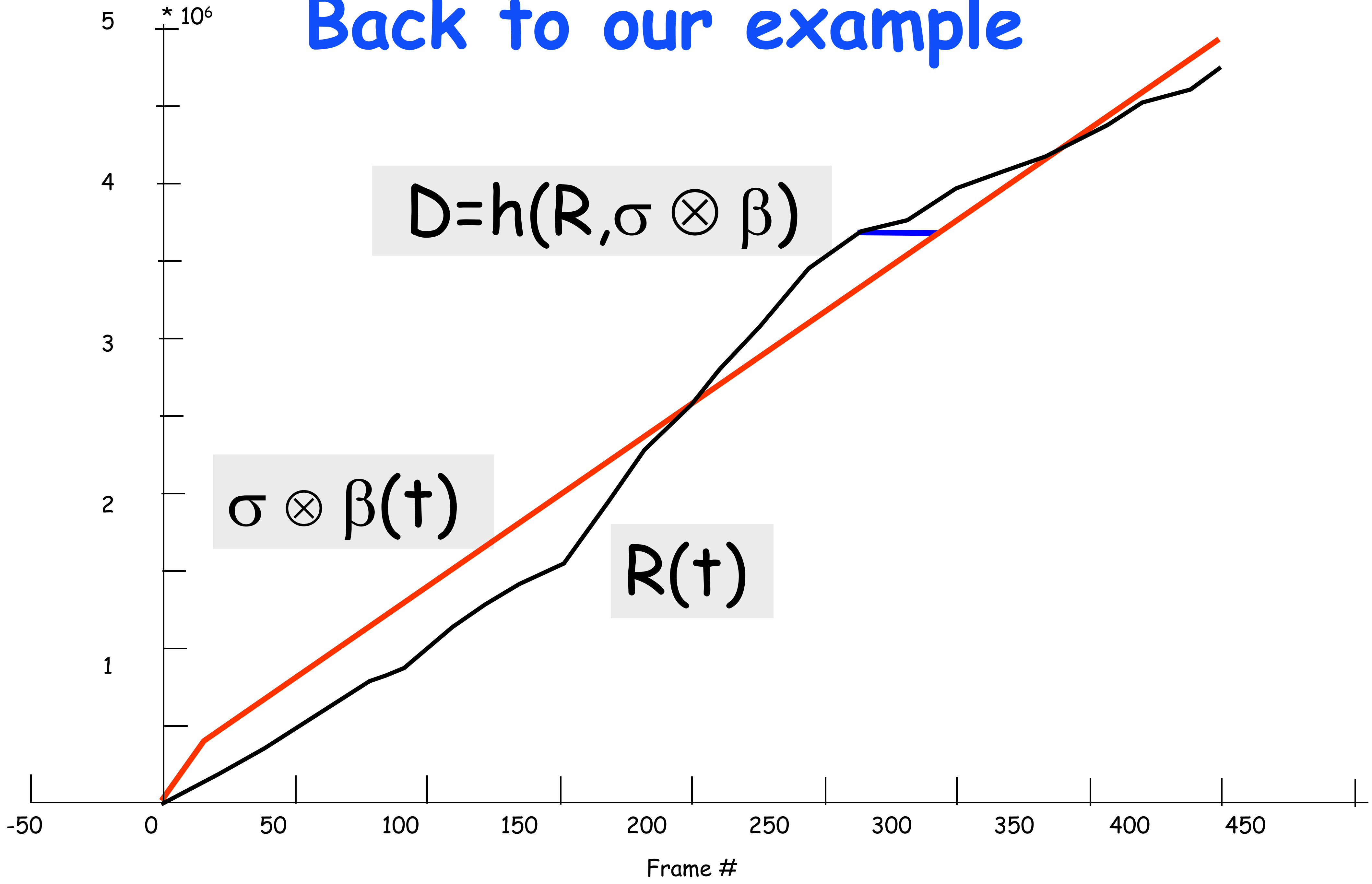
Example

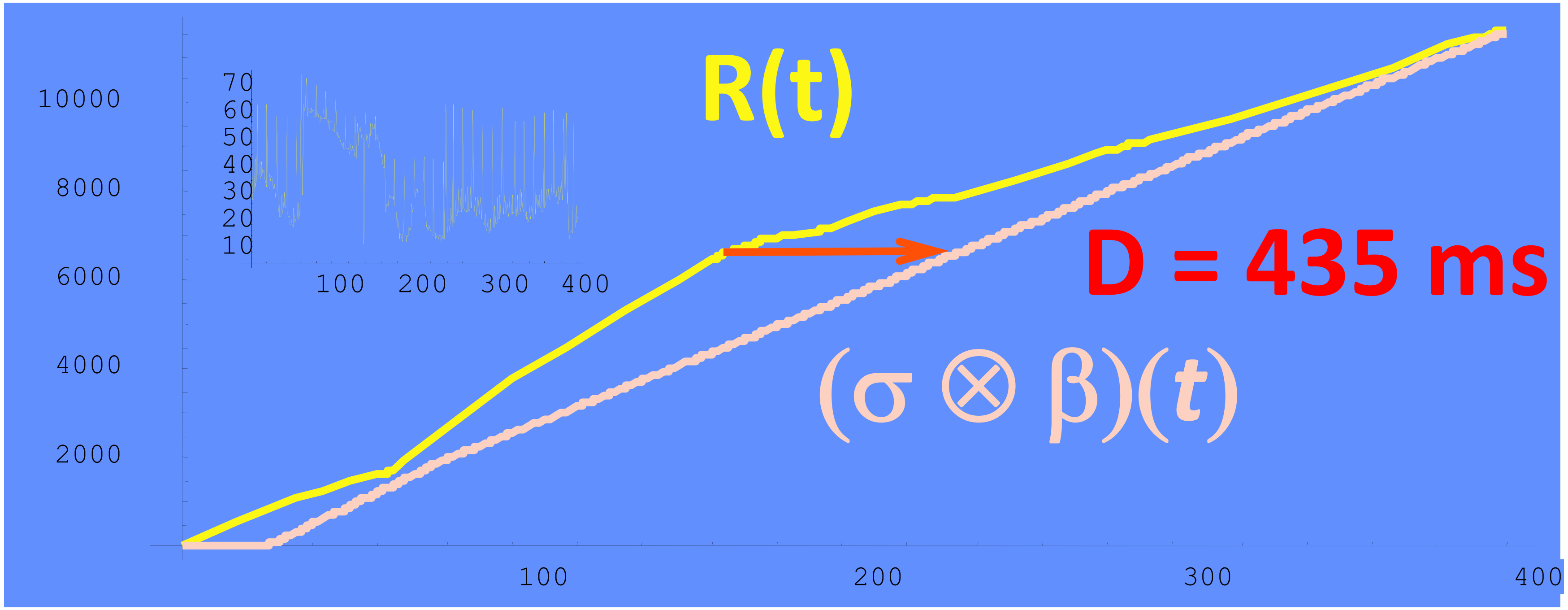

* 10^6
5
4
3
2
1
-50
250
300
350
400
450
$\sigma \otimes \beta(t)$
$R \oslash (\sigma \otimes \beta)$
$R(t)$
minimum value of D

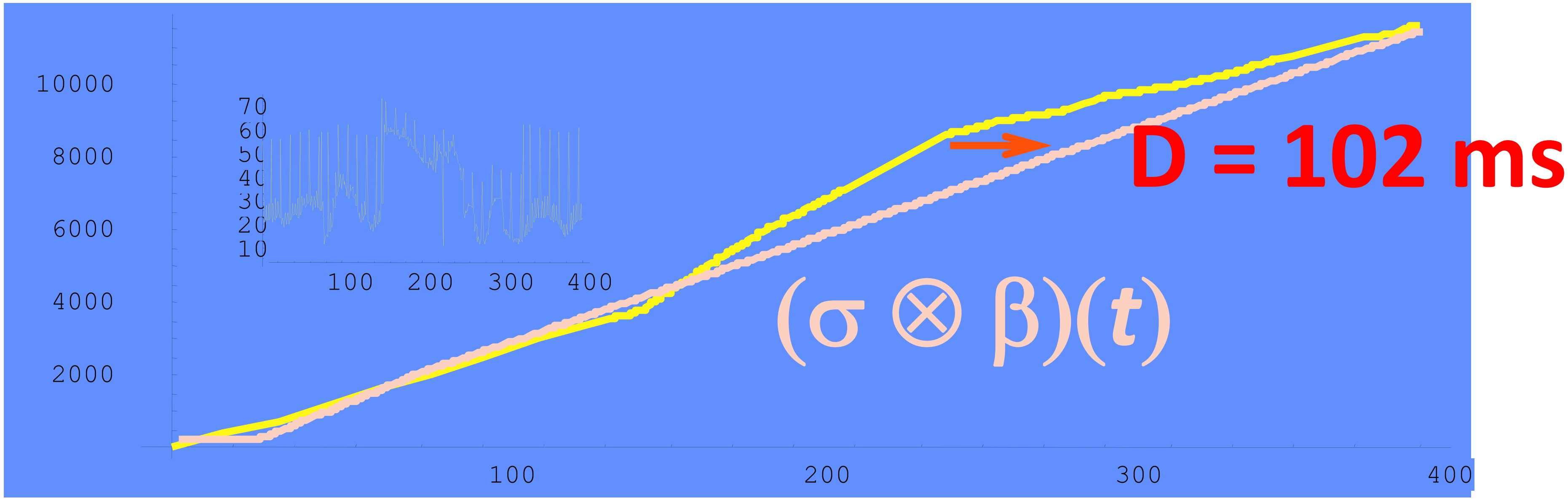
Minimum Playback Delay

- D must satisfy :
  $$R \oslash (\beta \otimes \sigma)(-D) \geq 0$$
- this is equivalent to
  $$D \geq h(R, \beta \otimes \sigma)$$

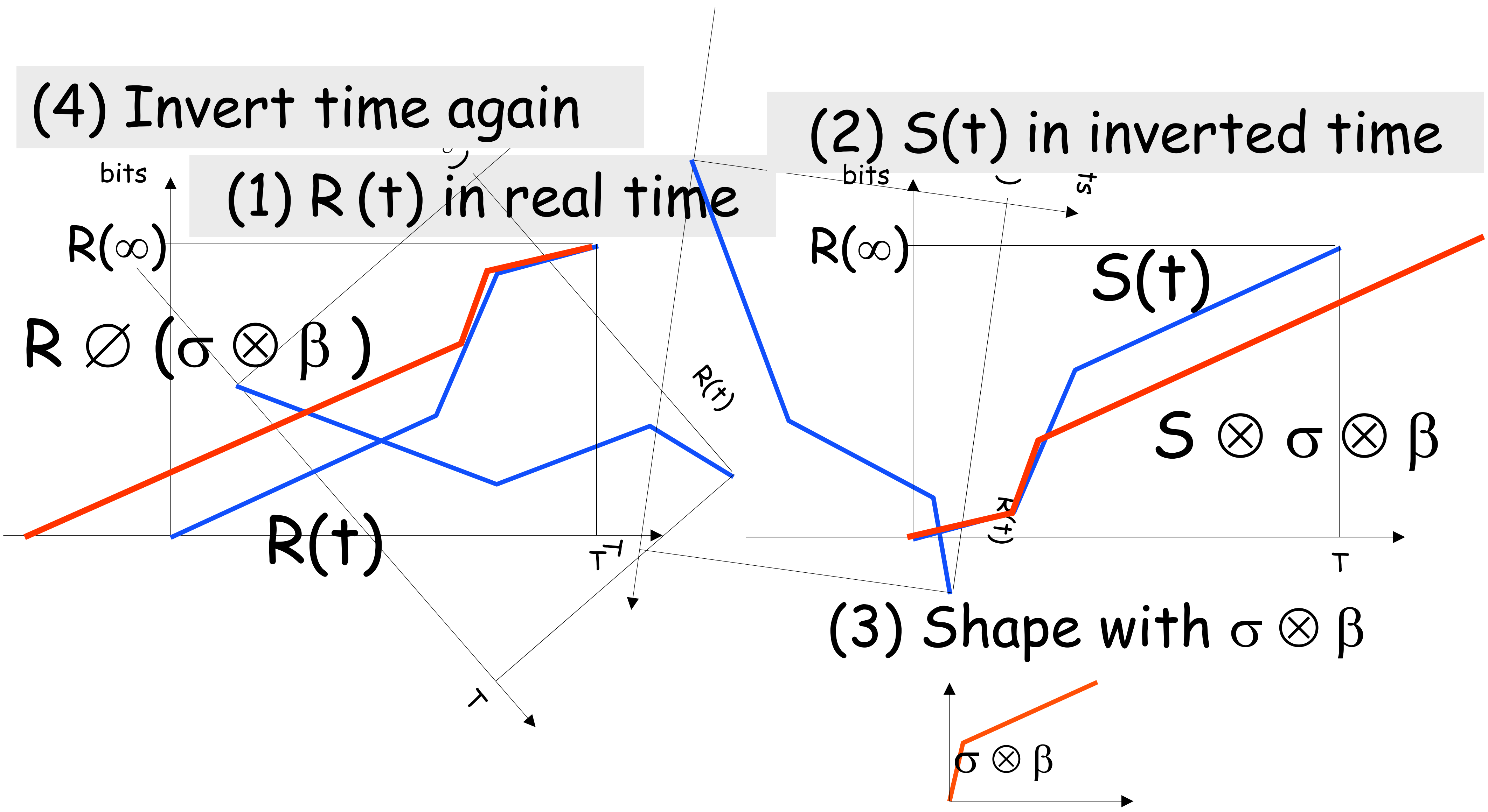
Back to our example




R(t)
D = 435 ms
(σ ⊗ β)(t)
10000
8000
6000
4000
2000
100
200
300
400
70
60
50
40
30
20
10
100 200 300 400

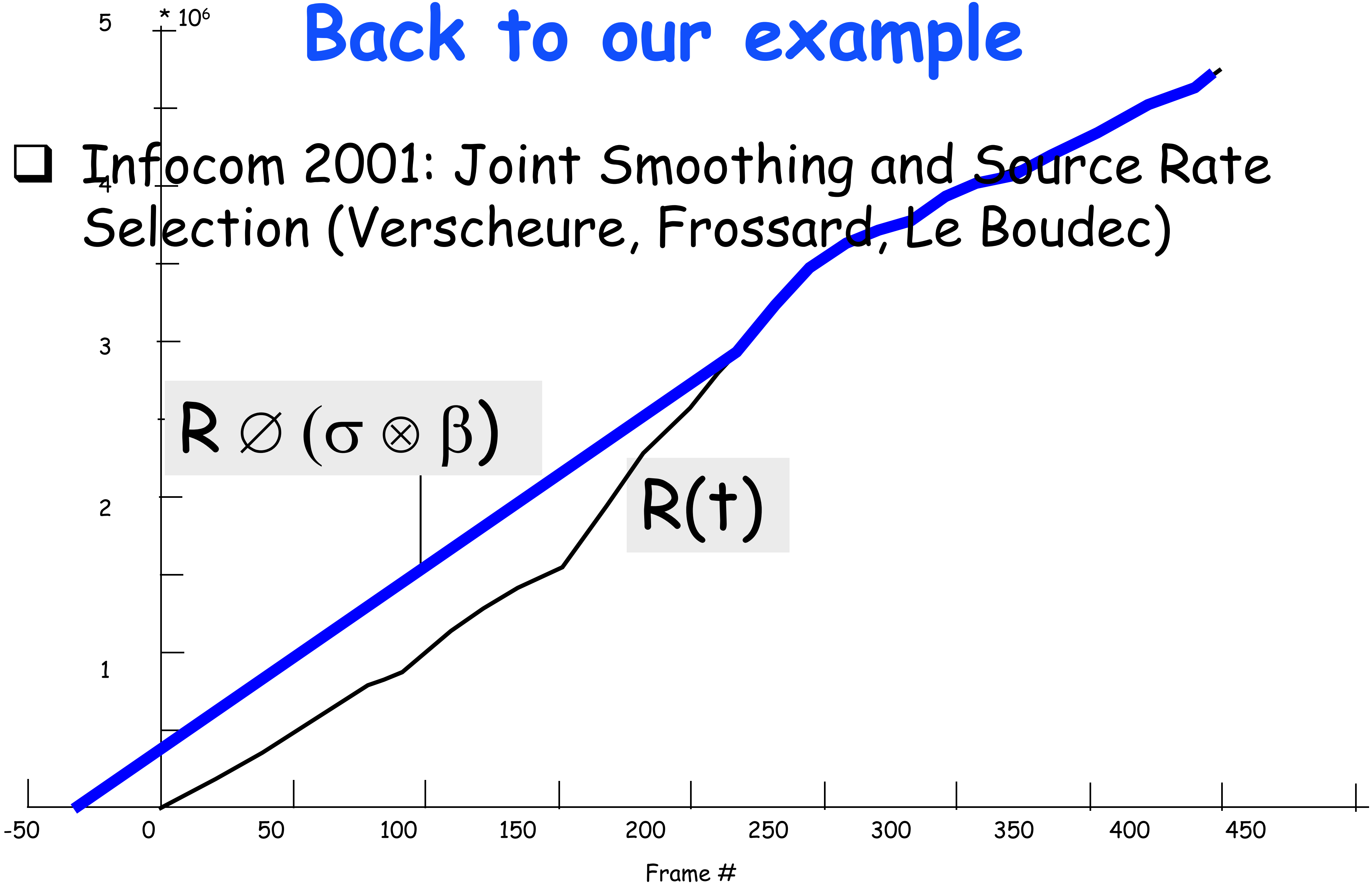

D = 102 ms
(σ ⊗ β)(t)
10000
8000
6000
4000
2000
100
200
300
400
70
60
50
40
30
20
10
100 200 300 400

Deconvolution is the time inverse of convolution



Back to our example

- Infocom 2001: Joint Smoothing and Source Rate Selection (Verscheure, Frossard, Le Boudec)



Conclusion

- Network Calculus is a set of tools and theories for the deterministic analysis of communication networks
- Application of min-plus algebra
- Does not supersede stochastic queueing analysis, but gives new tools for analysis of sample paths
- Book and slides available online at Le Boudec's or Thiran's home page