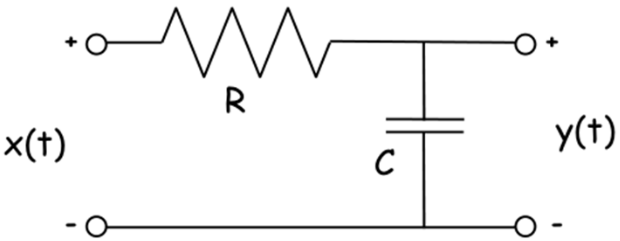
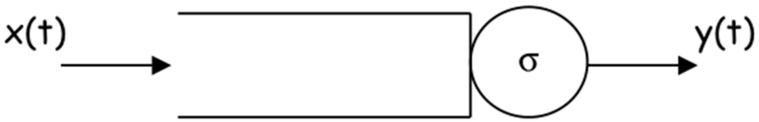
THE SYSTEM THEORY OF NETWORK CALCULUS



(a)



(b)

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EPFL

WoNeCa, 2012 Mars 21

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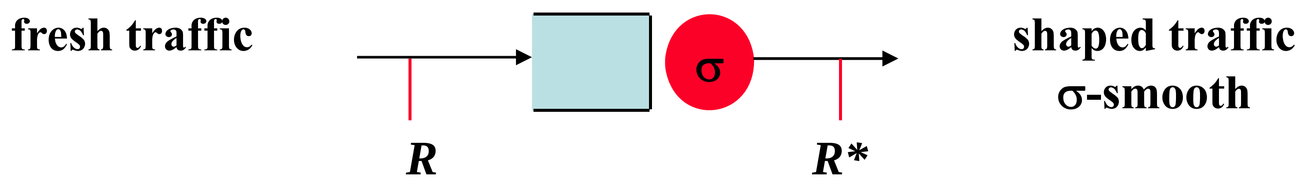
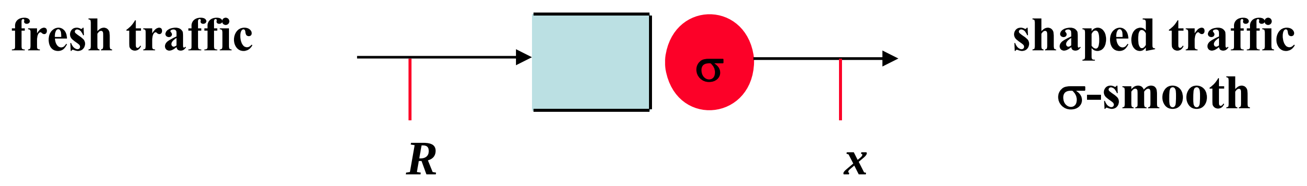
The Shaper



- shaper: forces output to be constrained by σ
- greedy shaper stores data in a buffer only if needed
- examples:
  - constant bit rate link (σ(t)=ct)
  - ATM shaper; fluid leaky bucket controller
- Pb: find input/output relation

A Min-Plus Model of Shaper



- Shaper Equations:

  (1) $x \leq x \otimes \sigma$

  (2) $x \leq R$

- R and x are functions
- σ is sub-additive
- $\otimes$ is min-plus convolution

Network Calculus's System Theory

- G = set of functions $Z \to R^+$ that are wide-sense increasing
- Also works in continuous time, functions are left-continuous $R \to R^+$
- An operator Π is a mapping : G -> G
- Π is isotone if $x(t) \leq y(t) \Rightarrow \Pi(x)(t) \leq \Pi(y)(t)$
- Π is upper-semi continuous iff $\inf_i(\Pi(x_i)) = \Pi(\inf_i(x_i))$ for $\downarrow$ sequences x_i

Min-Plus Linear Operators

- Π is min-plus linear if
  - for any constant K, $\Pi(x + K) = \Pi(x) + K$
    $$\Pi(x \wedge y) = \Pi(x) \wedge \Pi(y)$$
  - Π is upper-semi continuous.
- Representation Theorem: Π is min-plus linear <=> there is a unique H: R x R -> $\underline{R}^+$, ↑ in t and ↓ in s, such that $\Pi(x)(t)=\inf_s[H(t,s)+x(s)]$
- min-plus linear => isotone and upper semi-continuous
- Example: convolution operator
  $$C_\sigma:\ x \mapsto \sigma \otimes x$$
- Example: $M \in G$ is given:
  $$h_M:\ \ x \mapsto y \text{ s.t. } y(t) = \inf_{s \leq t}\big(M(t) - M(s) + x(s)\big)$$

Min-Plus Residuation Theorem

Theorem: ([L., Thiran 2001] thm 4.3.1., derived from Baccelli et al.,) Assume that Π is *isotone* and *upper-semi-continuous*. The problem

$$x(t) \le b(t) \wedge \Pi(x)(t)$$

where $x \in G$ is the unknown function

has one maximum solution in G, given by

$$x^*(t) = \underline{\Pi}(b)(t)$$

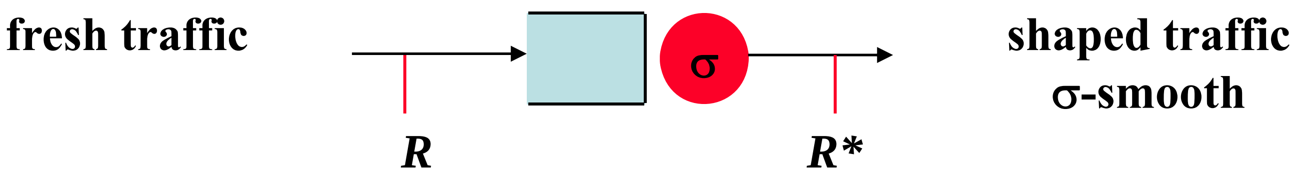
- (Definition of closure)

  $$\underline{\Pi}(x) = \inf\{x, \Pi(x), \Pi \circ \Pi(x), \Pi \circ \Pi \circ \Pi(x), \ldots\}$$

- in other words:
  $x^0 = b$; $x^i = \Pi(x^{i-1})$ and $x^* = \inf\{x^0, x^1, \ldots, x^i, \ldots\}$

$$(1)\ x \le x \otimes \sigma$$
$$(2)\ x \le R$$

Application to Shaper



- There is a maximum solution obtained by iterating

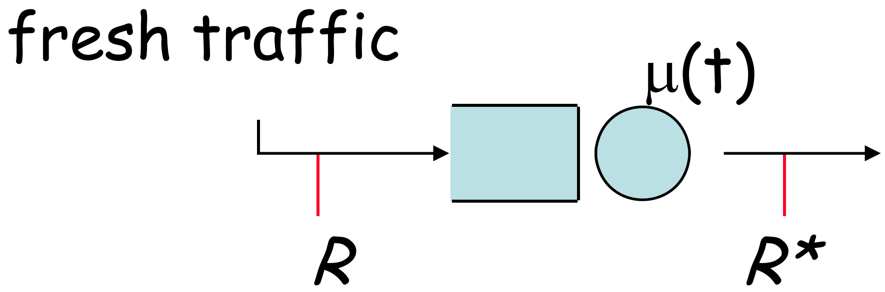
$$x^{(0)} = R$$
$$x^{(1)} = R \otimes \sigma$$
$$x^{(2)} = (R \otimes \sigma) \otimes \sigma = R \otimes \sigma$$

  because $\sigma \otimes \sigma = \sigma$

  Thus $R^* = \inf\left(x^{(0)}, x^{(1)}, x^{(2)}, \ldots\right) = R \otimes \sigma$

- The greedy shaper output is $R^* = R \otimes \sigma$
- $C_\sigma \circ C_\sigma = C_\sigma$, the subadditive closure of C_σ is C_σ

Variable Capacity Node



- node has a time varying capacity $\mu(t)$
  Define $M(t) = \int_0^t \mu(s)\, ds$.
- the output satisfies

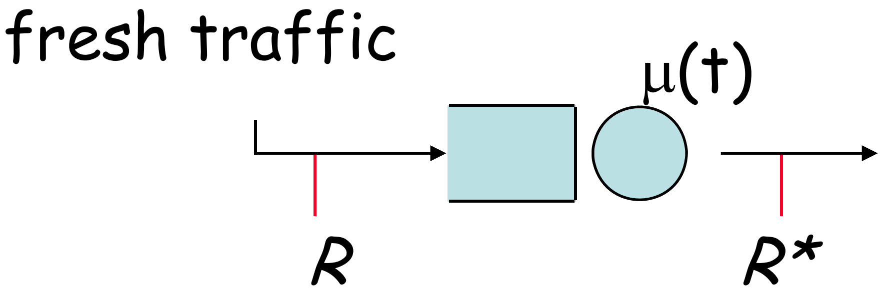
  $R^*(t) \leq R(t)$

  $R^*(t) - R^*(s) \leq M(t) - M(s)$ for all $s \leq t$

  and is "as large as possible"

Variable Capacity Node

R*(t) ≤ R(t)
R*(t) -R*(s) ≤ M(t) -M(s)
for all s ≤ t



- Operator h_M: $x \mapsto y$ s.t.

$$y(t) = \inf_{s \le t} M(t) - M(s) + x(s)$$

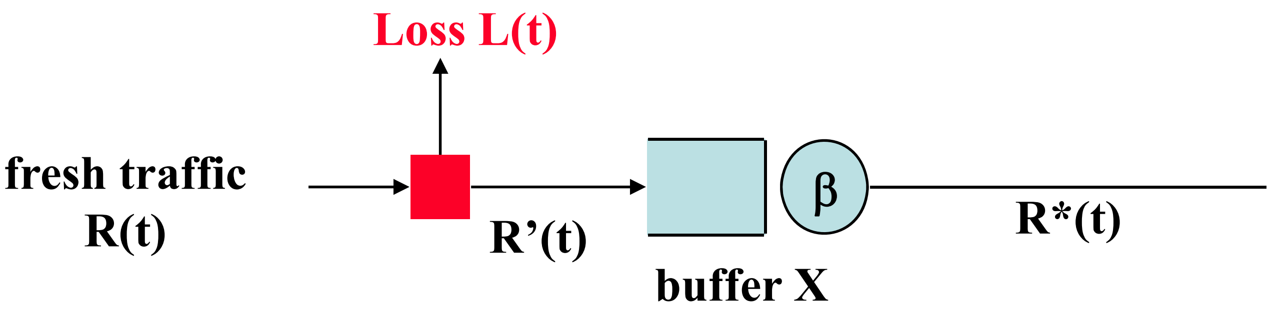
- We have the problem $R^* \le R, R^* \le h_M(R)$
- $h_M \circ h_M = h_M$ and the sub-additive closure of h_M is h_M
- There is a maximum solution,

$$R^*(t) = \inf_{s \le t} \big(M(t) - M(s) + R(s)\big)$$

2.

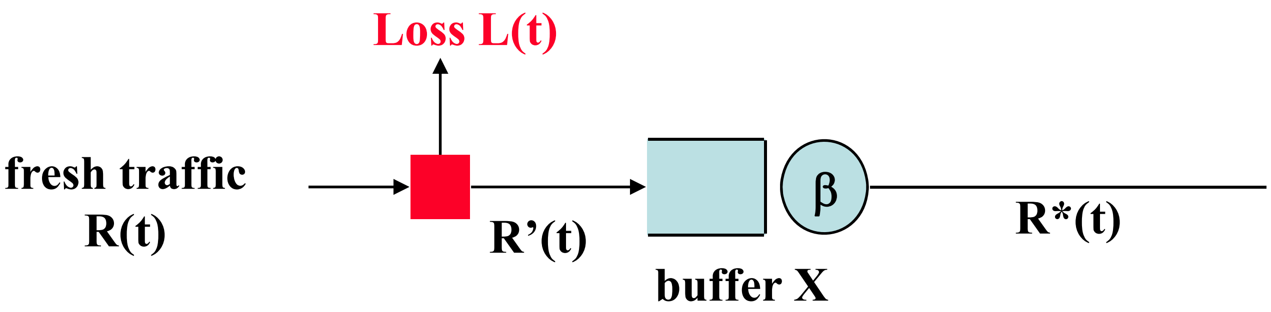
MORE EXAMPLES

A System with Loss [Chuang and Cheng 2000]

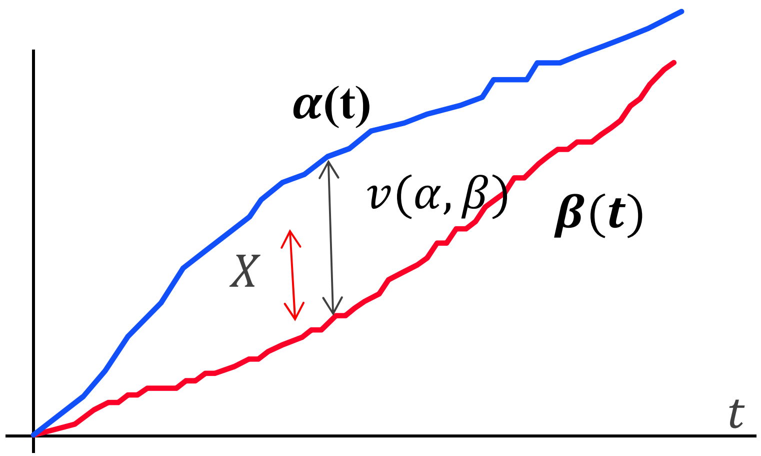


- node with service curve $\beta(t)$ and buffer of size X
- when buffer is full incoming data is discarded
- modelled by a virtual controller (*not* buffered)
- fluid model or fixed sized packets
- Pb: find loss ratio

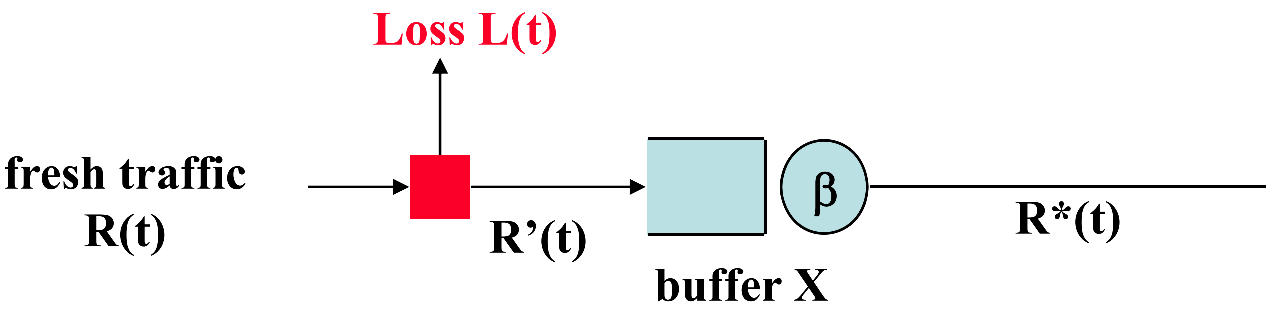
A System with Loss



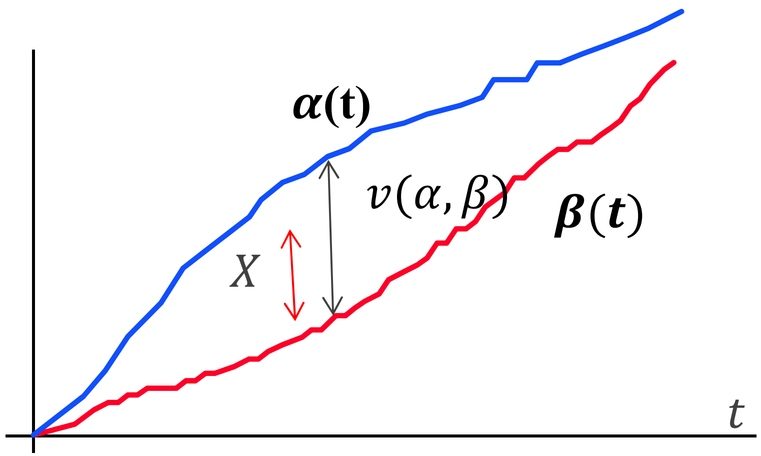
- Assume R is $\alpha-$smooth; if $X \geq v(\alpha,\beta)$ then no loss
- If $X < h(\alpha,\beta)$, what can we say ?

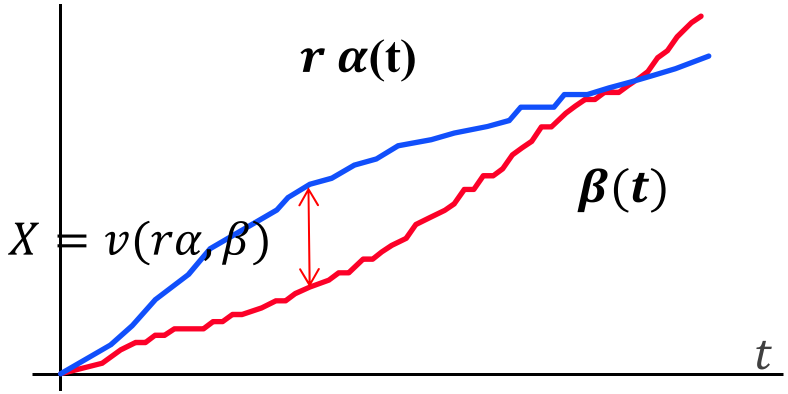


Bound on Loss Ratio

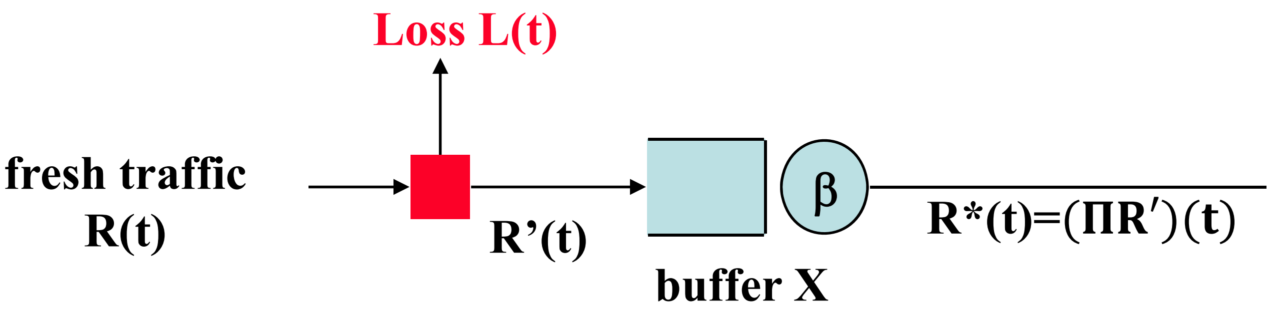


- **Thm** [Chuang and Cheng 2000] Let r be the largest such that $X = v(r\alpha, \beta)$ i.e. $r = 1 \wedge \inf_{t>0} \left(\frac{\beta(t)+X}{\alpha(t)} \right)$
  Then $\frac{L(t)}{R(t)} \le 1 - r$; it is the best possible bound.





Analysis of System with Loss

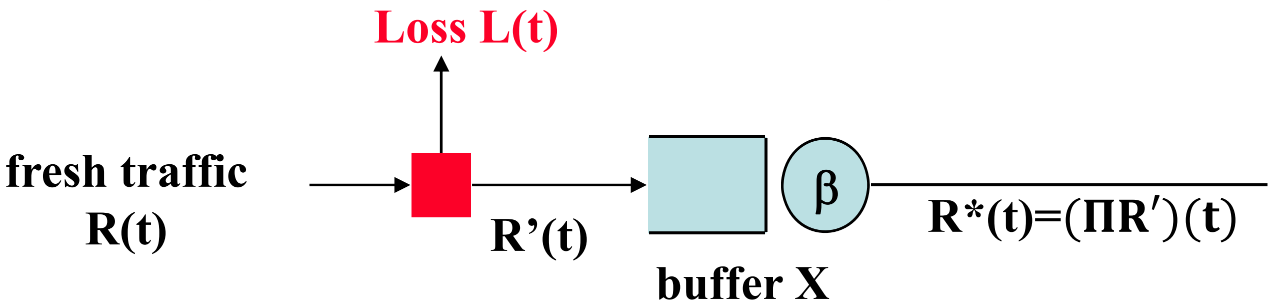


1. $R'(t) - R'(s) \leq R(t) - R(s)\ \forall\, s \leq t$ (splitter)
2. $R'(t) - \Pi R'(t) \leq X$ (buffer does not overflow) where Π is the transformation R' -> R, assumed isotone and usc (« physical assumptions »)

- There is a maximum solution and R' is the maximum solution

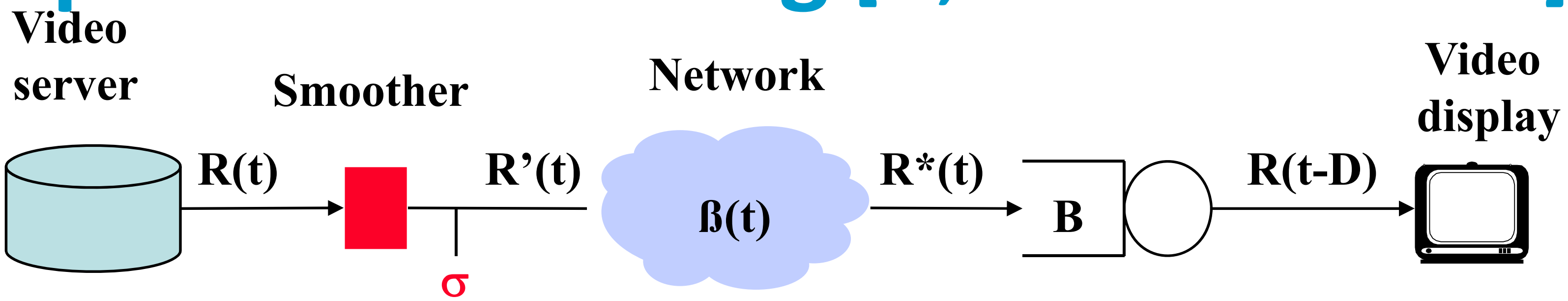
Analysis of System with Loss

1. $R'(t) - R'(s) \leq R(t) - R(s) \ \forall\, s \leq t$ (splitter)
2. $R'(t) - \Pi R'(t) \leq X$ (buffer does not overflow)



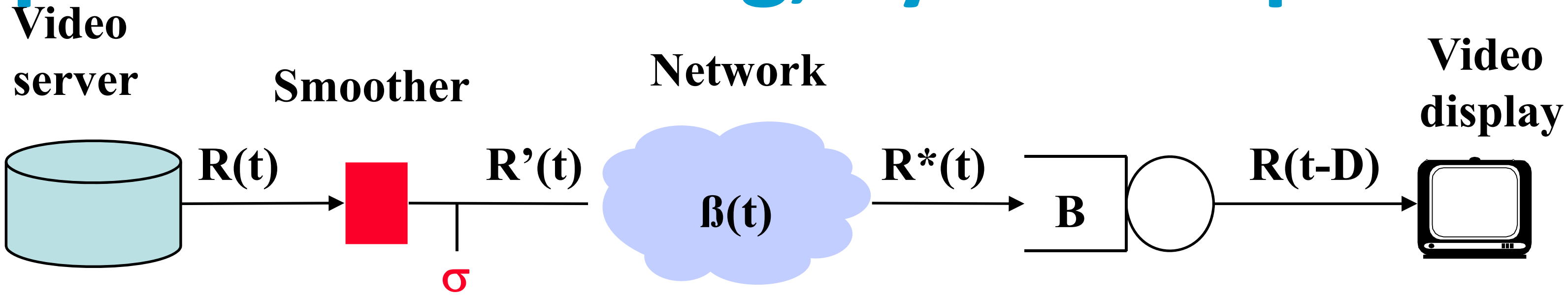
- Let $x(t) = rR(t)$ with r given by thm.
- Eqn 1 is satisfied
- x is $r\alpha-$smooth, thus required buffer $\leq X$ and Eqn 2 is satisfied
- Thus $R'(t) \geq x(t)$ and
  $\frac{L(t)}{R(t)} = 1 - \frac{R'(t)}{R(t)} \leq 1 - \frac{x(t)}{R(t)} = 1 - r$

Optimal Smoothing [L.,Verscheure 2000]



- Network + end-client offer a service curve β to flow $R'(t)$
- Smoother delivers a flow $R'(t)$ conforming to an arrival curve σ.
- Video stream is stored in the client buffer, read after a playback delay D.
- Pb: which smoothing strategy minimizes D?

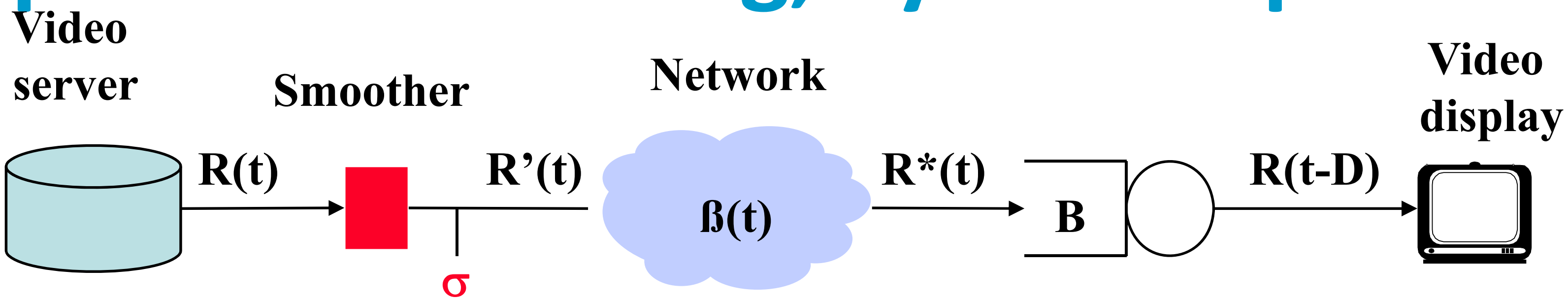
Optimal Smoothing, System Equations



- (1) R' is σ-smooth
  (2) $(R' \otimes \beta)(t) \geq R(t-D)$
- Use deconvolution $(a \oslash b)(t) = \sup_{s \geq 0} (a(t+s) - b(s))$

  $x \leq y \otimes \beta \iff x \oslash \beta \leq y$
- system becomes
  (1) $R' \geq R' \oslash \sigma$
  (2) $R' \geq (R \oslash \beta)(t-D)$

Optimal Smoothing, System Equations

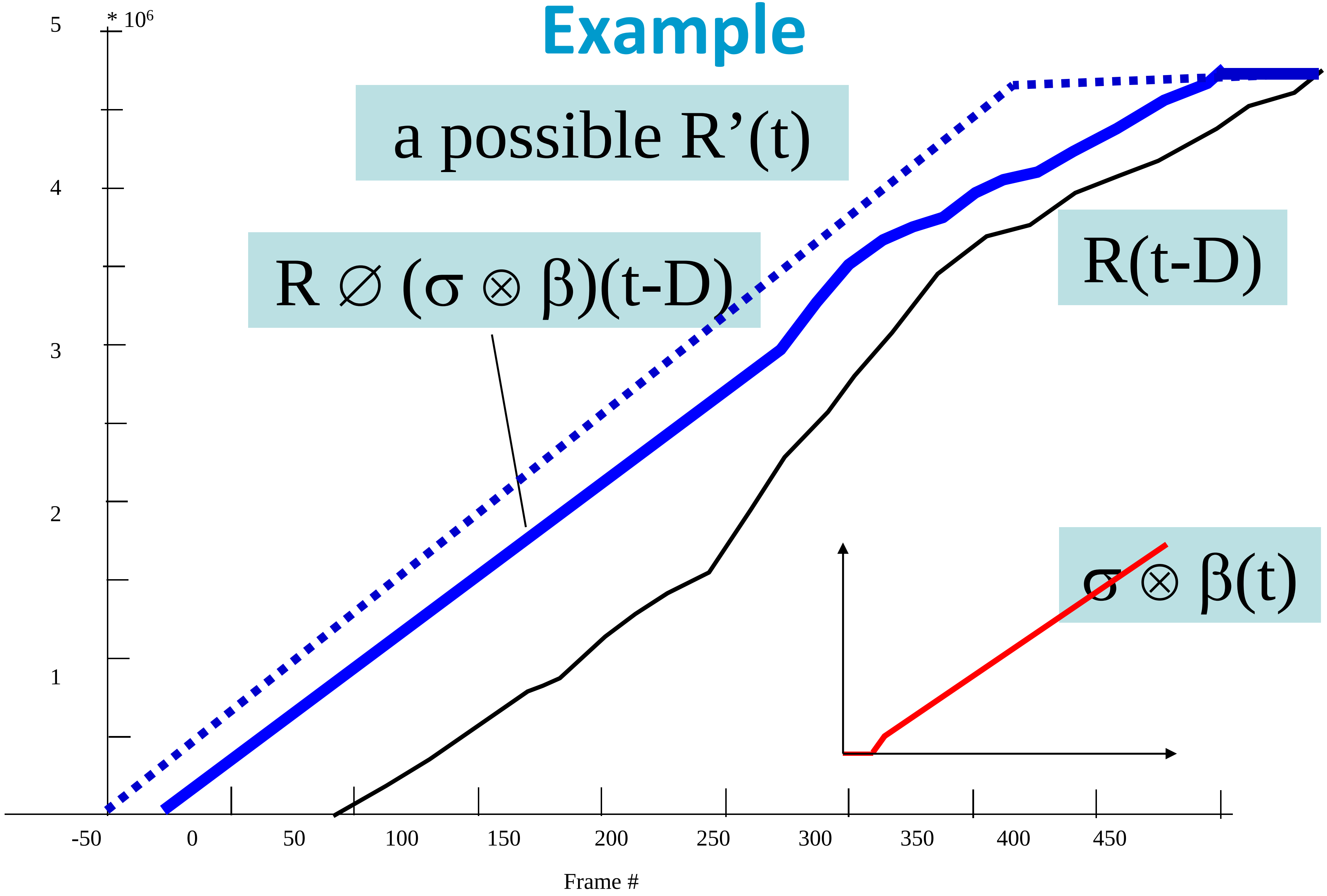


(1) $R' \geq R' \oslash \sigma$
(2) $R' \geq (R \oslash \beta)(t\text{-}D)$

- This is a max-plus linear problem, it has a minimum solution R' given by the iterations:
  $x^{(0)}(t) = (R \oslash \beta)(t - D)$
  $x^{(1)}(t) = x^{(0)} \oslash \sigma\,(t) = (R \oslash (\sigma \otimes \beta))(t\text{-}D)$
  $x^{(2)}(t) = x^{(1)}(t)$ because $\sigma \oplus \sigma = \sigma$

- Thus $R'(t) = (R \oslash (\sigma \otimes \beta))(t-D)$

Example



* 10^6
5
4
3
2
1
a possible R'(t)
R ∅ (σ ⊗ β)(t-D)
R(t-D)
σ ⊗ β(t)
-50
0
50
100
150
200
250
300
350
400
450
Frame #

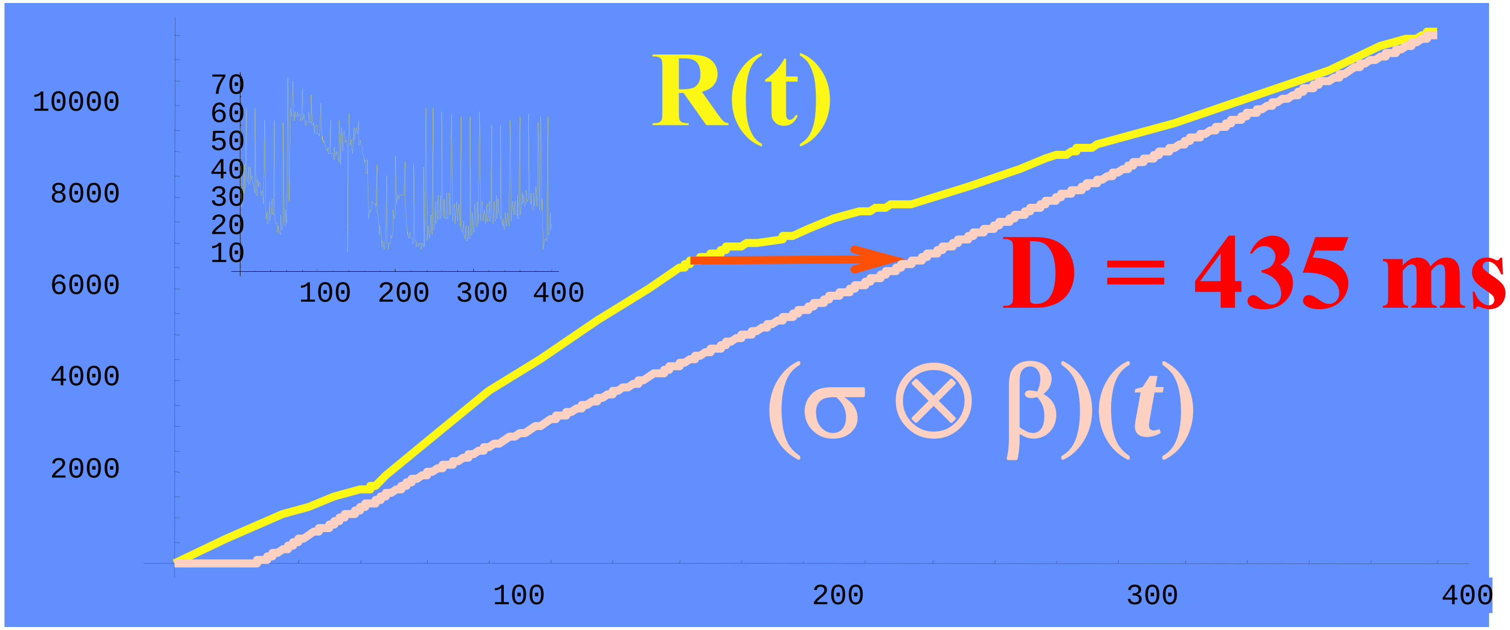
Minimum Playback Delay

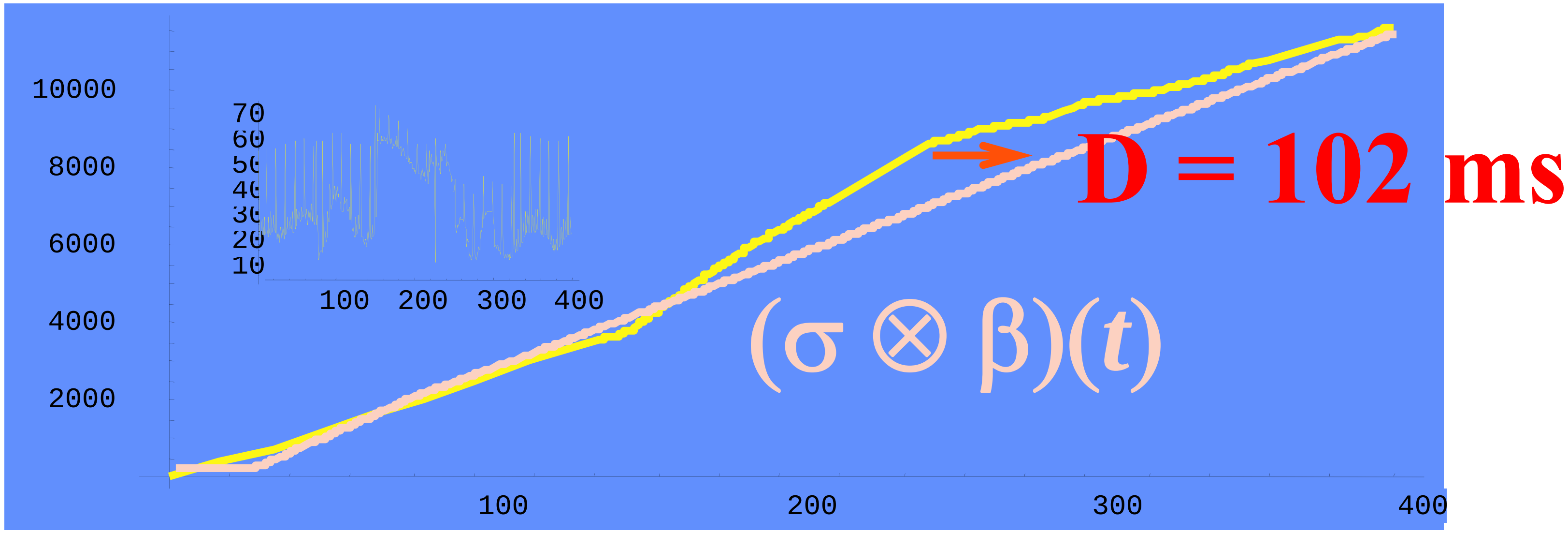
- D must satisfy :

  $$R \oslash (\beta \otimes \sigma)\,(-D) \geq 0$$

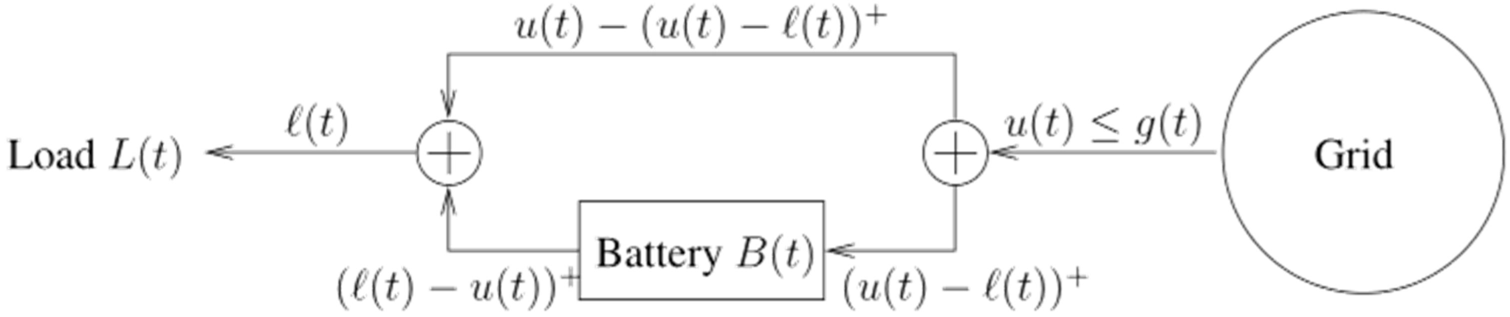
- this is equivalent to

  $$D \geq h(R, \beta \otimes \sigma)$$


R(t)
D = 435 ms
(σ ⊗ β)(t)
10000
8000
6000
4000
2000
100
200
300
400
70
60
50
40
30
20
10

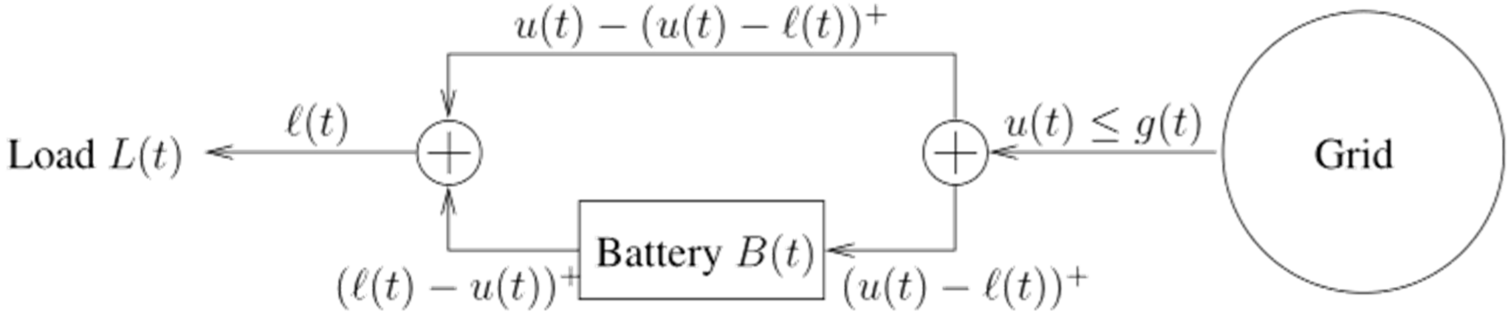

D = 102 ms
(σ ⊗ β)(t)
10000
8000
6000
4000
2000
100
200
300
400

The Perfect Battery



- Battery may be charged $(u(t) > \ell(t))$or discharged $(u(t) < \ell(t))$
- Load $\ell(t)$ is given
- Problem is to determine a power schedule $u(t)$, subject to $0 \leq u(t) \leq g(t)$ and within battery constraints

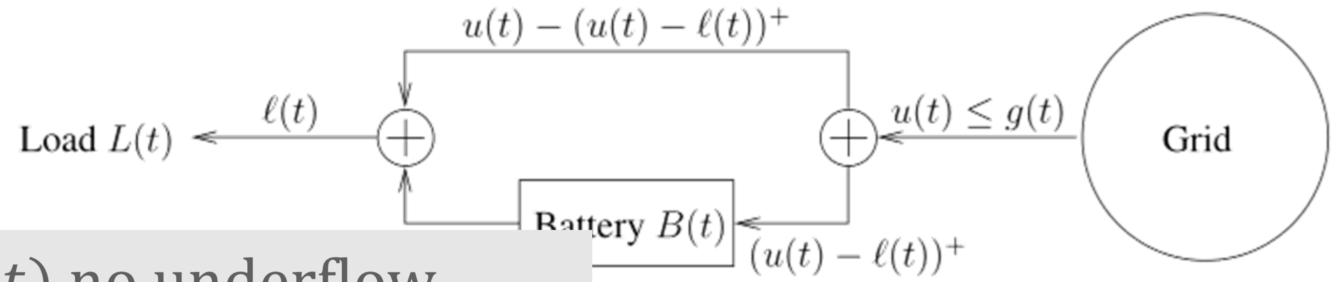
System Equations for the Perfect Battery



1. $L(t) \leq B_0 + U(t)$ no underflow
2. $U(t) - L(t) + B_0 \leq B$ no overflow
3. $U(t) - U(s) \leq G(t) - G(s), \forall s \leq t$ power constraint

where $U(t), L(t), G(t)$ are cumulative functions such as $U(t) = \int_0^t u(s)ds$

System Equations



1. $L(t) \le B_0 + U(t)$ no underflow
2. $U(t) - L(t) + B_0 \le \; B$ no overflow
3. $U(t) - U(s) \le G(t) - G(s), \forall s \le t$

- Relax (eq 1):

$$U(t) \le \big(B - B_0 + L(t)\big)1_{t>0}$$

$$U(t \le h_G(U)(t)$$

There is a maximum solution,

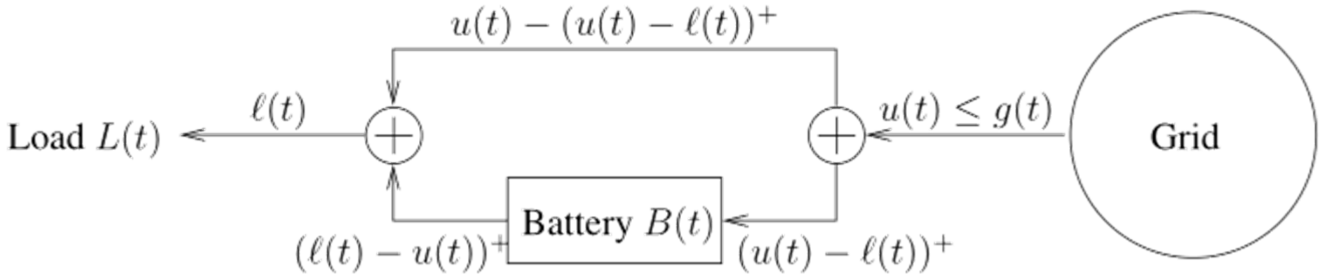
$$U^*(t) = G(t) \wedge \inf_{s \le t}(G(t) - G(s) + L(s) + B - B_0)$$

U^* is causal

The problem is feasible iff U^*satisfies (eq 1), i.e.

$$\begin{cases} B_0 \ge \sup_t \big(L(t) - G(t)\big) \\ B \ge \sup_{0 \le s \le t} (L(t) - L(s) - G(t) + G(s)) \end{cases}$$

System Equations



1. $L(t) \le B_0 + U(t)$ no underflow
2. $U(t) - L(t) + B_0 \le B$ no overflow
3. $U(t) - U(s) \le G(t) - G(s), \forall s \le t$

- Relax (eq 2):

$$U(t) \ge \max(0, -B_0 + L(t))$$
$$U(s) \ge \sup_{\tau \ge s} (G(s) - G(\tau) + U(\tau))$$

There is a minimum solution,

$$U_*(t) = 0 \vee \sup_{\tau \ge t} (G(t) - G(\tau) + L(\tau) - B_0)$$

U_* is non-causal

The problem is feasible iff U_* satisfies (eq 2)

This gives the same conditions

3.

TIME VERSUS SPACE

The Residuation Theorem is a Space Method

- The maximum solution x^* to the problem

$$x \leq b$$
$$x \leq \Pi x$$

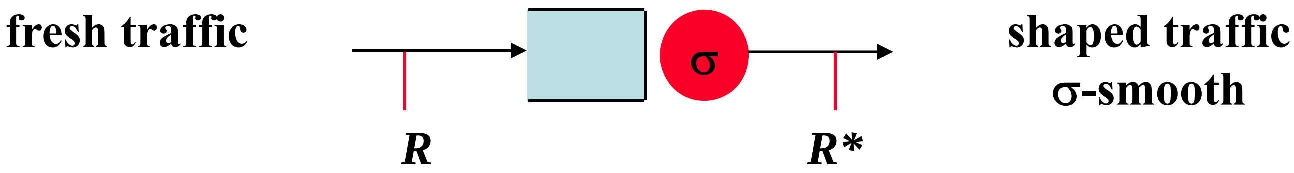
  is given by iterates over the entire trajectory

$$x^{(0)} = b$$
$$x^{(1)} = \Pi x^{(0)}$$
$$x^{(2)} = \Pi x^{(1)}$$
$$etc$$

- When time is discrete there may be another way to compute x^* by time recursion

$$\boxed{\begin{array}{c}(1)\ x \le x \otimes \sigma \\ (2)\ x \le R\end{array}}$$

The Shaper, Time Method



- Time is discrete $t = 0,1,2,\ldots$
- Define R' by:

$$R'(0) = R(0) = 0$$
$$R'(t) = R(t) \wedge \inf_{0 \le u \le t-1} (\sigma(t-u) + R'(u))$$

- R' is solution
- For any other solution x, $x(t) \le R'(t)$ [induction]
- R' is the maximal solution, i.e. $R' = R^*$.
- Note the difference in representation:

$$R^*(t) = R(t) \wedge \inf_{0 \le u \le t-1} \big(\sigma(t-u) + R(u)\big)$$

The Time Method for *Linear* Problems

- [L., Thiran 2001] Thm 4.4.1: the problem in discrete time

$$x(t) \leq b(t)$$
$$x(t) \leq \inf_s \left(H(t,s) + x(s)\right)$$

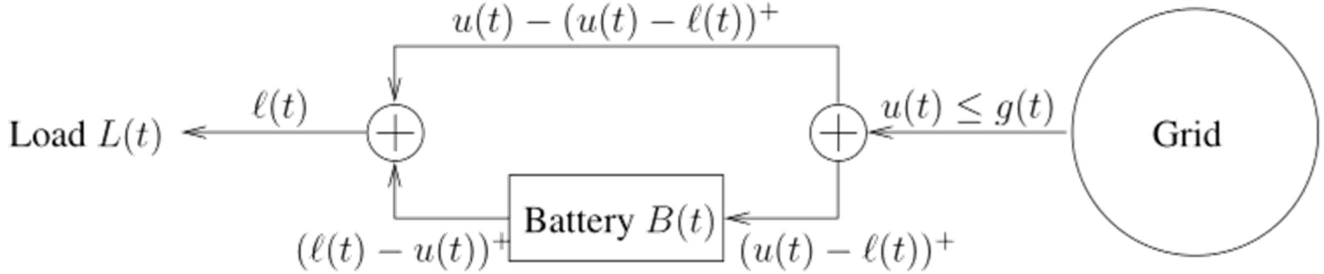
where $H : N \times N \to R^+$, $\uparrow$ in t and $\downarrow$ in s

has a maximal solution x^* given by

$$x^*(0) = b(0)$$
$$x^*(t) = x(t) \wedge \inf_{0 \leq u \leq t-1} \left(H(t,u) + x^*(u)\right)$$

- This is a second, alternative representation for x^*

Perfect Battery



There is a maximum solution,

$$U^*(t) = G(t) \wedge \inf_{s \leq t} (G(t) - G(s) + L(s) + B - B_0)$$

It can be computed by the time method:

$$u^*(t) = \min\,(g(t), B - B(t) + \ell(t))$$

The minimum schedule is anti-causal and can be computed with time reversal

Conclusion

- Min-plus and max-plus system theory contains a central result : residuation theorem (= fixed point theorem)
  Establishes existence of maximum (resp. minimum) solutions
  and provides a representation

- Space and Time methods give different representations

Thank You...