Mean Field Methods for Computer and Communication Systems: A Tutorial

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Introduction

The method presented in this tutorial finds its roots in fluid limits of statistical physics (where it is called interacting particles), and was successfully applied in contexts such as communication and computer system modelling [5], biology [6] or game theory [2]. It can be applied to the analysis and simulations of systems with many objects; its main features are the simplification of the global model by a fluid limit, while retaining a stochastic model for an individual object. Below is a quick summary of the tutorial.

1 Mean Field Interaction Model

We consider a generic model of N interacting objects, where each object has a state and interaction between objects is Markovian, i.e. the evolution of the system depends only on the collection of states at any point in time. The state description is such that objects can be observed only through their state. This is quite general, and allows heterogeneous settings with different classes of objects [1].

We need a scaling assumption, which describes how the system evolves when N gets large. We assume that there is some well defined intensity $I(N)$ such that the number of transitions undergone by one object per time unit when N is large is of the order of $I(N)$. Two frequent cases of interest are: (1) when $I(N)$ is constant and (2) when $I(N)$ vanishes (i.e. $\lim_{N\to\infty} I(N) = 0$, for example $I(N) = 1/N$).

2 Finite Horizon Results

2.1 The Mean Field Limit

When N gets large, under very general assumptions (discussed in the next section), there is convergence of the empirical measure of the global process to a deterministic system, which we call the "mean field limit". The mean field limit may be an Ordinary Differential Equation as in [1] or a discrete time, iterated map as in [17].

A remarkable associated fact is "propagation of chaos". A generic theorem [19] states that, whenever convergence to the mean field limit holds, the states of any finite numbers of objects are asymptotically independent, and distributed according to the mean field limit. Thus, the mean field limit enjoys the dual property of being both an approximation to the deterministic limit of the global system *and* to the state distribution of one (or any finite number of) objects. This is essential, and explains why mean field theory is much more than a deterministic approximation.

2.2 Convergence Results

There are generic results for a large class of powerful models. Essentially, when the state space per object is finite, or even enumerable, then convergence to mean fields follows from simple scaling assumptions and structural properties verifiable by inspection [15, 18, 4, 1, 17, 21]. When the state space is not enumerable, the situation is more open [12, 11, 10]. We also discuss some examples with more general frameworks [14].

These convergence results are over a finite horizon. Essentially, they state that the mean field limit is a good approximation over any fixed, finite horizon, rescaled by the intensity.

2.3 Random Process Modulated by Mean Field Limit

This is a generalization of propagation of chaos, which states that, in the mean field limit, the state of one object can be approximated by a Markov process modulated by the mean field limit [8]. This provides a powerful method to perform fast analysis or fast simulation.

3 Stationary Regime

The mean field convergence results are for finite horizons. In practice, one is often interested in the stationary behaviour of the system. Fortunately, the mean field limit is useful here too, however one needs to be careful how to use it.

3.1 Stationary Behaviour of Mean Field Limit

Assume that the system with finite N has a unique stationary distribution (a frequent case) and that the mean field limit is an ODE, say $\dot{y} = F(y)$. Since $y(t)$ is an approximation of Y^N, a natural approximation of the stationary distribution of Y^N is the set of stationary points of the ODE, obtained by solving $F(y) = 0$. Assume we find a unique solution y^*, then one is tempted to say that y^* is the mean field approximation of the stationary distribution of Y^N, and therefore, also of the distribution of states of one object. While this sometimes holds, it must not always be so, as we show with some example. Thus, obtaining the mean field limit is not all, one should also "*study the ODE*".

3.2 A Critique of the Fixed Point Approximation

The fixed point approximation consists in solving $F(y) = 0$ in the mean field limit and assuming that the obtained solution is an approximation of the state distribution for one object. This is justified by the propagation of chaos result only when the stationary behaviour of the mean field limit is indeed obtained in this way (as assumption that may or may not hold, depending on the ODE). Otherwise, objects may become dependent in the stationary regime. For example, in [7] it is shown that the fixed point assumption does not hold for some parameter settings of a wireless system analyzed in [3], due to limit cycles in the fluid limit. In such cases, the mean field limit may reveal important qualitative features of the system, such as synchronization, which may be hard to obtain by classical methods.

3.3 The Reversible Case

There is a class of systems for which such complications may not arise, namely the class of reversible stochastic processes. Reversibility is classically defined as a property of time reversibility in stationary regime [13]. For example, the stochastic process Y^N of [14], which describes the occupancy of inter-city telecommunication links, is reversible. In such cases, the fluid limit must have stationary points, and any limit point of the stationary distribution of Y^N must be supported by the set of stationary points [16]. Thus, for reversible processes that have a fluid limit, the fixed point approximation is justified.

4 Optimization

The mean field limit may be used to analyze optimization problems on the model, both under centralized (markov decision processes) and decentralized (game theoretic) settings. Under general assumptions, the limit of the optimal value function for the model with finite N is equal to the optimal value function of the mean field limit. This gives ways to address problems that are too difficult at finite N due to combinatorial explosion [20, 9].

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