Mean Field Methods for Computer and Communication Systems
Part 2: Infinite Horizon

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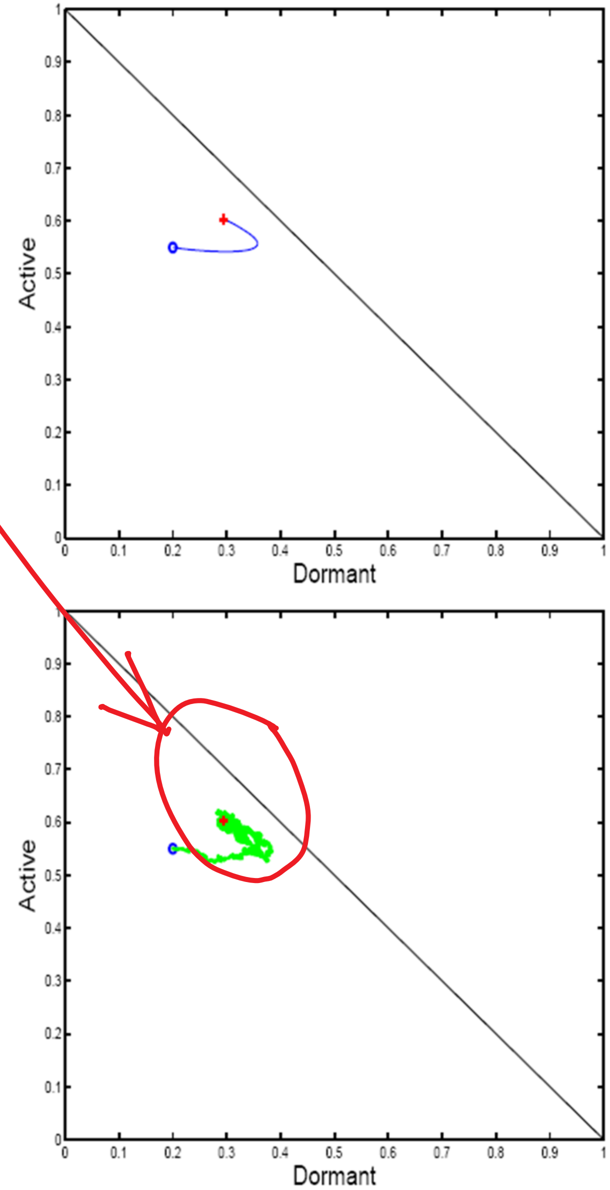
STATIONARY REGIME, FIXED POINT AND THE DECOUPLING ASSUMPTION

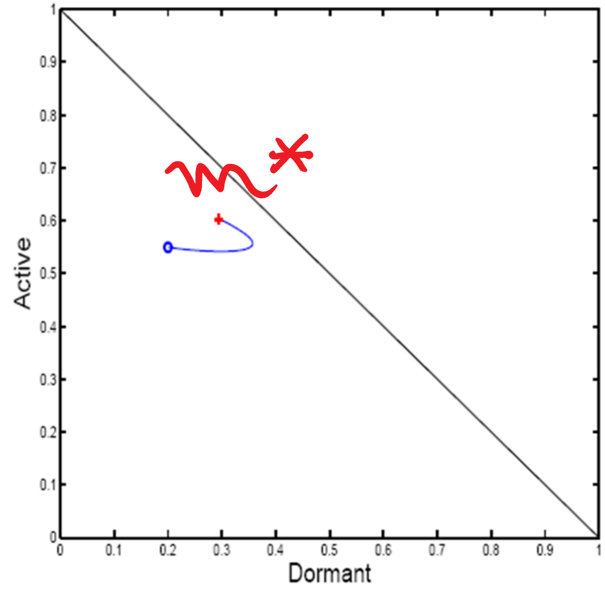
Stationary regime = for large t

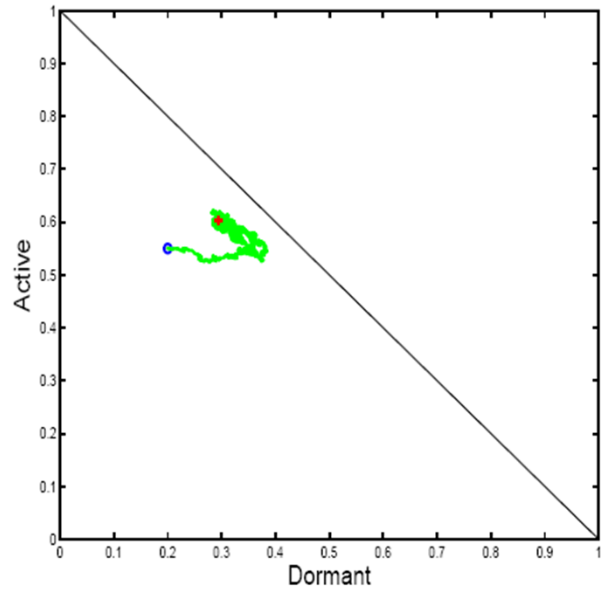
- The mean field limit suggests that
  - Prob (node n is dormant) ≈ 0.3
  - Prob (node n is active) ≈ 0.6
  - Prob (node n is susceptible) ≈ 0.1
- Decoupling assumption says distribution of prob for state of one object is $\approx \vec{m}(t)$ with $\frac{d\vec{m}(t)}{dt} = F(\vec{m}(t))$
- We are interested in stationary regime, i.e we do $F(\vec{m}) = 0$



The Fixed Point Method

- Assume a mean field appoximation
  $\frac{d\vec{m}(t)}{dt} = F(\vec{m}(t))$
- Let m^* be a solution of $F(m^*) = 0$
  (m^* is called a fixed point)
- Assume the system with finite N is ergodic (has a unique stationary distribution) π^N
- The fixed point method says that, for large N, $\pi^N \approx m^*$
- When is this valid ?





The Balance Equation

- We are looking for an approximation of the stationary proba π^N for the state of one node
- Balance Equation
  $\pi^N(i) \times (\text{proba of leaving } i) = \sum_j \pi^N(j) \times (\text{proba of reaching } i\ )$

case	prob
1	$D\delta_D$
2	$D\lambda\frac{ND-1}{N-1}$
3	$A\beta\frac{D}{h+D}$
4	$A\delta_A$
5	$S(\alpha_0 + rD)$
6	$S\alpha$

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
5. Recruitment by Dormant
   - S + D -> D + D

   Direct infection
   - S -> D
6. Direct infection
   - S -> A

- Here: (with $i = A$):
  Proba that a given A node leaves state A ?

- Proba that transition 4 is made is $A^N \delta_A$
  There are NA^N nodes in state A
  Proba that a given A node makes a transition 4 is $\frac{A^N \delta_A}{NA^N} = \frac{\delta_A}{N}$
- We obtain the balance equation

$$\pi^N(A)\frac{\delta_A}{N} = \pi^N(D)\frac{2}{ND^N}D^N\lambda\frac{ND^N - 1}{N-1} + \pi^N(D)\frac{1}{ND^N}A^N\beta\frac{D^N}{h+D^N} + \pi^N(S)\frac{1}{NS^N}S^N\alpha$$

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
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case	prob
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4	$A\delta_A$
5	$S(\alpha_0 + rD)$
6	$S\alpha$

- We obtain the balance equation

$$\pi^N(A)\delta_A = 2\pi^N(D)\lambda \frac{ND^N - 1}{N-1}$$
$$+ \pi^N(D)A^N\beta \frac{1}{h + D^N}$$
$$+\pi^N(S)\alpha$$

- Make the mean field approximation:

$$\pi^N(A) \approx A$$
$$A^N \approx A$$

- Obtain (with N large):

$$A\delta_A = 2D^2\lambda + \beta A \frac{D}{h+D} + S\alpha$$

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
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case	prob
1	$D\delta_D$
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3	$A\beta\frac{D}{h+D}$
4	$A\delta_A$
5	$S(\alpha_0 + rD)$
6	$S\alpha$

The Fixed Point Assumption is Equivalent to the Making the Mean Field Assumption (Decoupling Assumption) Proba for one object = Occupancy measure

- We obtained

$$A\delta_A = 2D^2\lambda + \beta A \frac{D}{h+D} + S\alpha$$

- This is one of the components of the fixed point equation $F(\vec{m}) = 0$

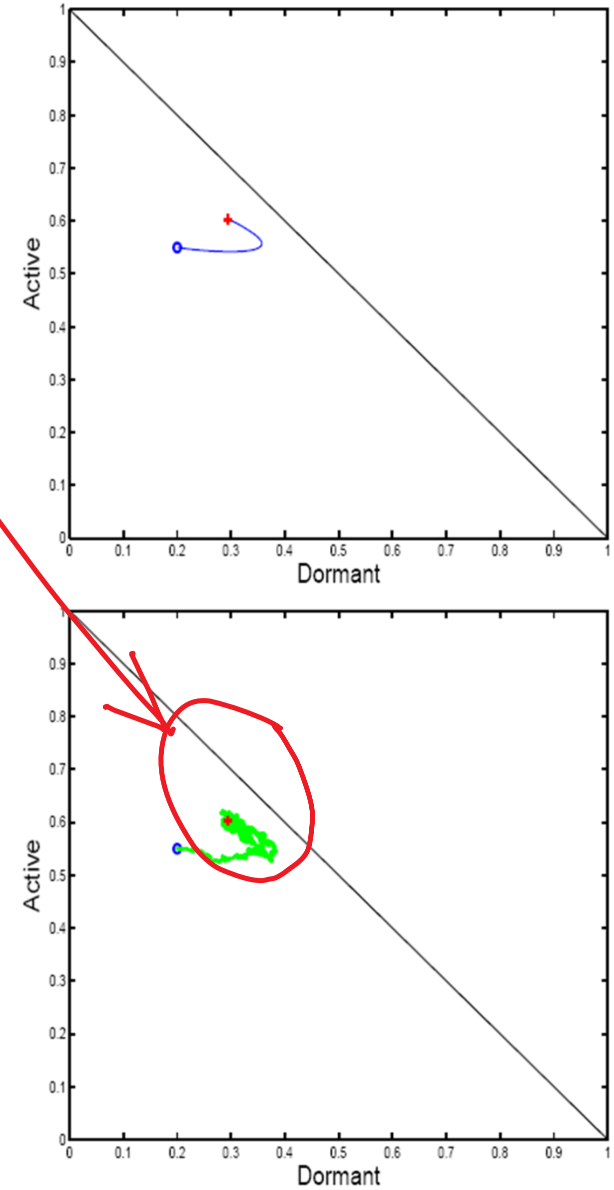
$$\begin{aligned}
\frac{\partial D}{\partial t} &= -\delta_D D - 2\lambda D^2 - \beta A \frac{D}{h+D} + (\alpha_0 + rD)S \\
\frac{\partial A}{\partial t} &= 2\lambda D^2 + \beta A \frac{D}{h+D} - \delta_A A + \alpha S \\
\frac{\partial S}{\partial t} &= \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S
\end{aligned}$$

ODE

Checkpoint

- The fixed point method finds the large N approximation of the state probabitiy for one object by solving $F(\vec{m}) = 0$

- This is the same as writing the balance equation and making the decoupling assumption in stationary regime



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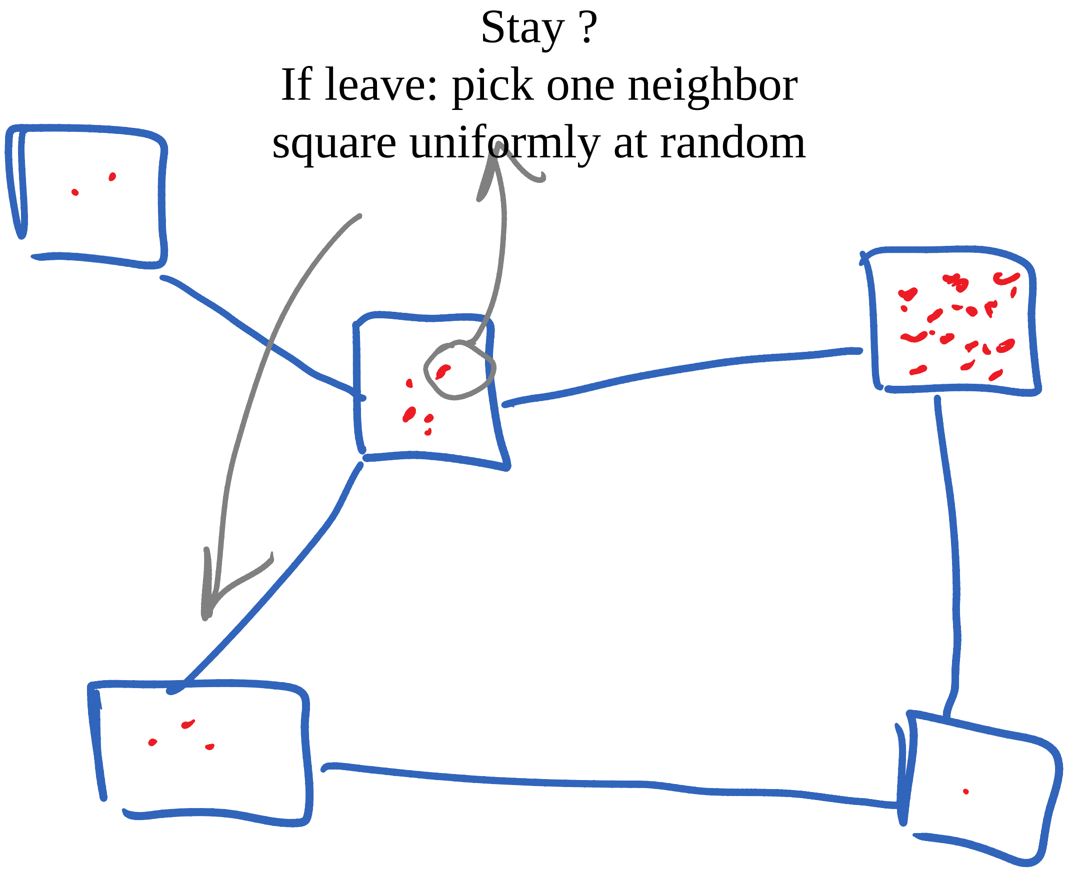
A CRITIQUE OF THE FIXED POINT METHOD

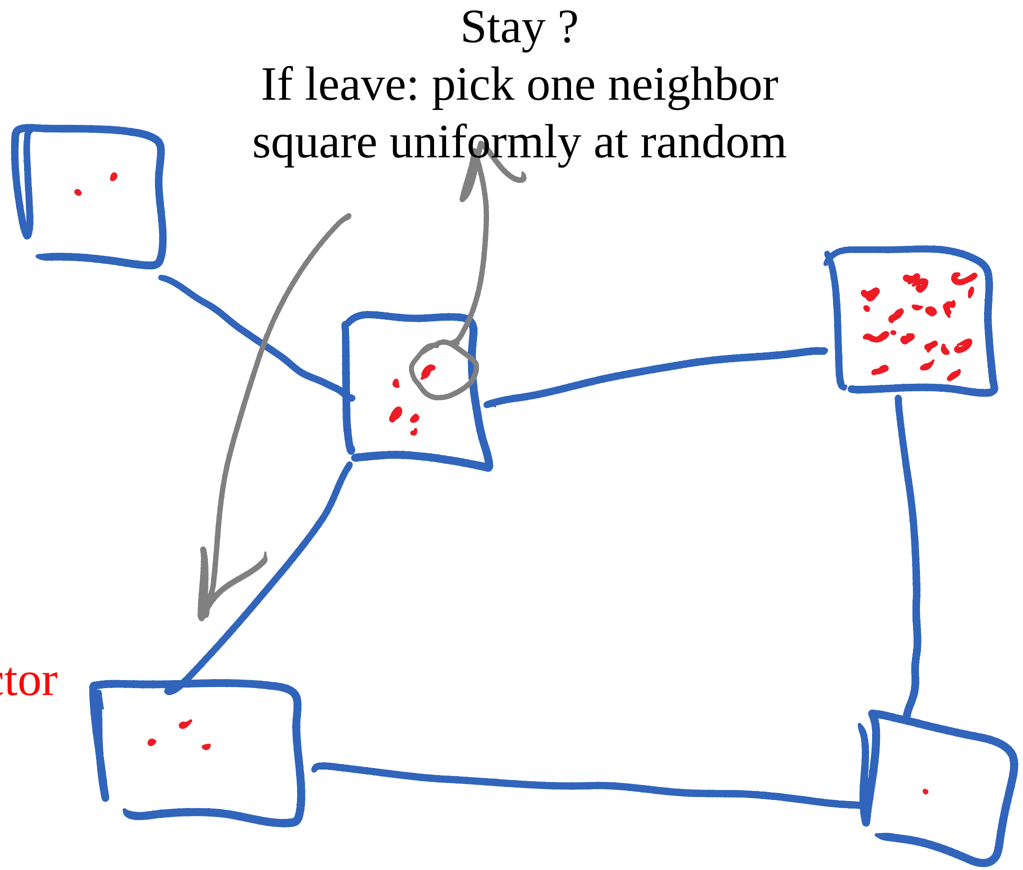
Another Example: El Botellon [Rowe 2003]

- Squares (with pubs) in a city
- From time to time, people move to another square
- Proba of movig depends on chat probability
- [Rowe 2003] shows emergence of concentration in one square



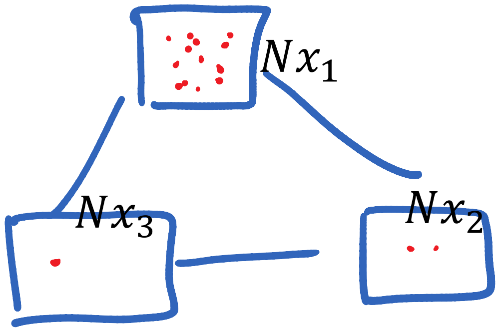
Mean Field Model of El Botellon [Bortolussi 2012]

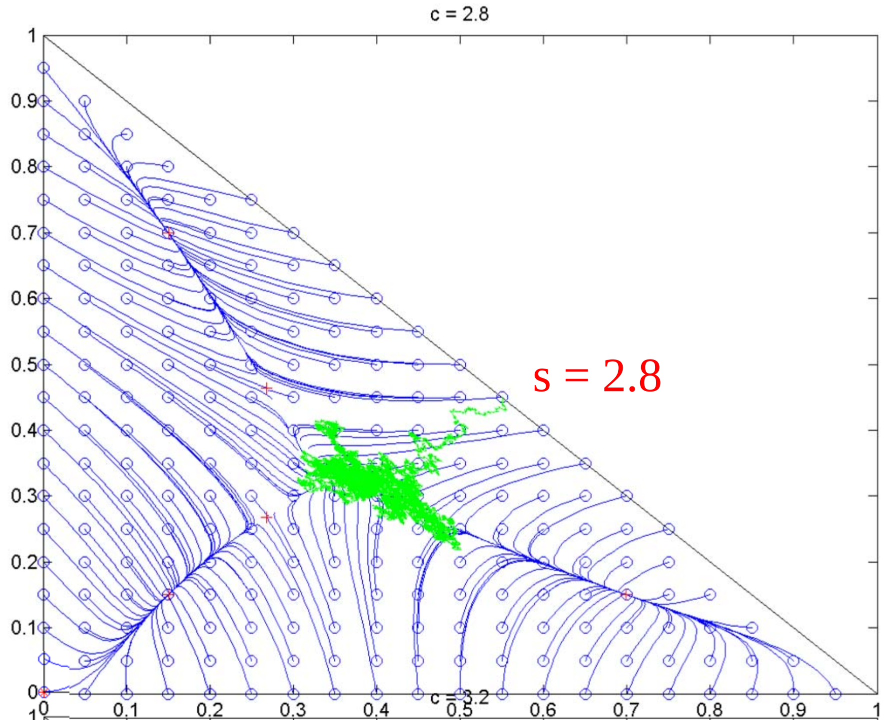
- N people in total, Nx_i are in square i
- At every time slot pick one person uniformly at random; Say she is in square i; proba this person leaves this square is $\left(1-\frac{s}{N}\right)^{Nx_i-1}$ Socialization factor

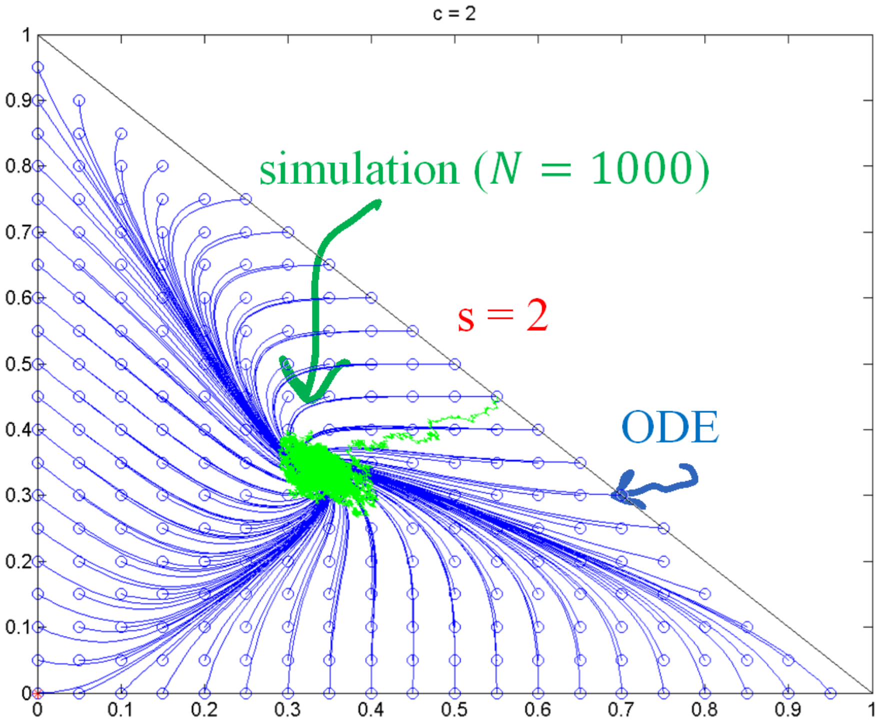


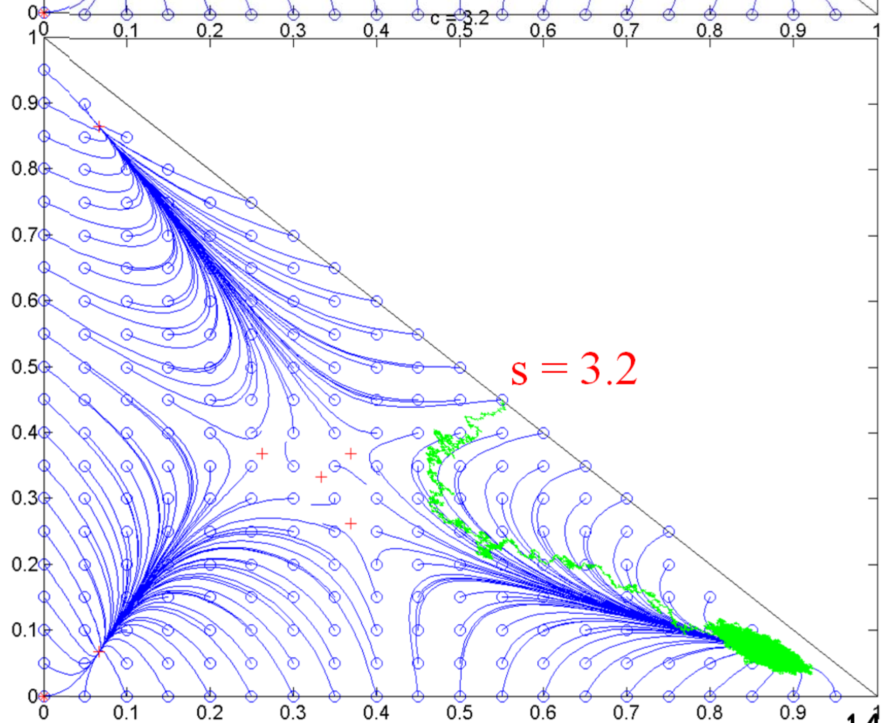
- There is convergence to mean field (1 transition per time slot)

3 Squares [Bortolussi 2012]

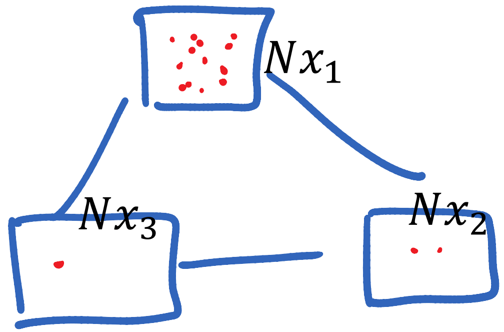






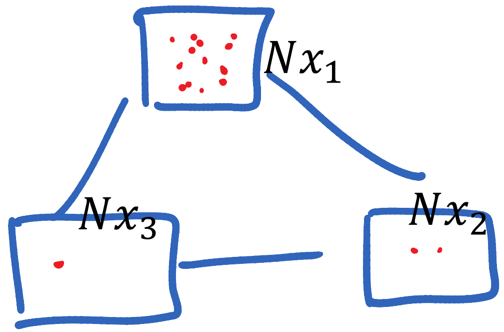


Mean Field Limit with 3 Squares [Bortolussi 2012]



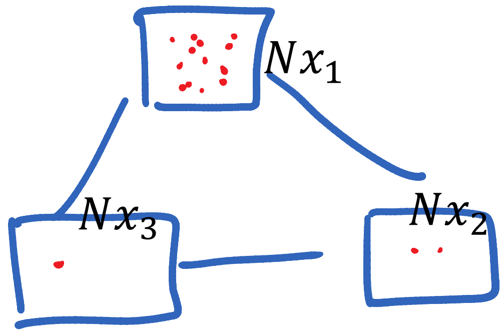
Transition	Proba	Delta to (x_1, x_2)
$1 \to 2$		
$1 \to 3$		
$2 \to 1$		
$2 \to 3$		
$3 \to 1$		
$3 \to 2$		

Mean Field Limit with 3 Squares [Bortolussi 2012]



Transition	Proba	Delta to (x_1, x_2)
$1 \rightarrow 2$	$x_1\left(1-\frac{s}{N}\right)^{Nx_1-1} \times \frac{1}{2}$	$\frac{1}{N}(-1,+1)$
$1 \rightarrow 3$	$x_1\left(1-\frac{s}{N}\right)^{Nx_1-1} \times \frac{1}{2}$	$\frac{1}{N}(-1,0)$
$2 \rightarrow 1$	$x_2\left(1-\frac{s}{N}\right)^{Nx_2-1} \times \frac{1}{2}$	$\frac{1}{N}(+1,-1)$
$2 \rightarrow 3$	$x_2\left(1-\frac{s}{N}\right)^{Nx_2-1} \times \frac{1}{2}$	$\frac{1}{N}(0,-1)$
$3 \rightarrow 1$	$(1-x_1-x_2)\left(1-\frac{s}{N}\right)^{N(1-x_1-x_2)} \times \frac{1}{2}$	$\frac{1}{N}(+1,0)$
$3 \rightarrow 2$	$(1-x_1-x_2)\left(1-\frac{s}{N}\right)^{N(1-x_1-x_2)} \times \frac{1}{2}$	$\frac{1}{N}(0,+1)$

Mean Field Limit with 3 Squares [Bortolussi 2012]



- The Mean Field limit is obtained by computing the drift and using
$$\lim_{N\to\infty}\left(1-\frac{s}{N}\right)^{Nx_i} = e^{-sx_i}$$

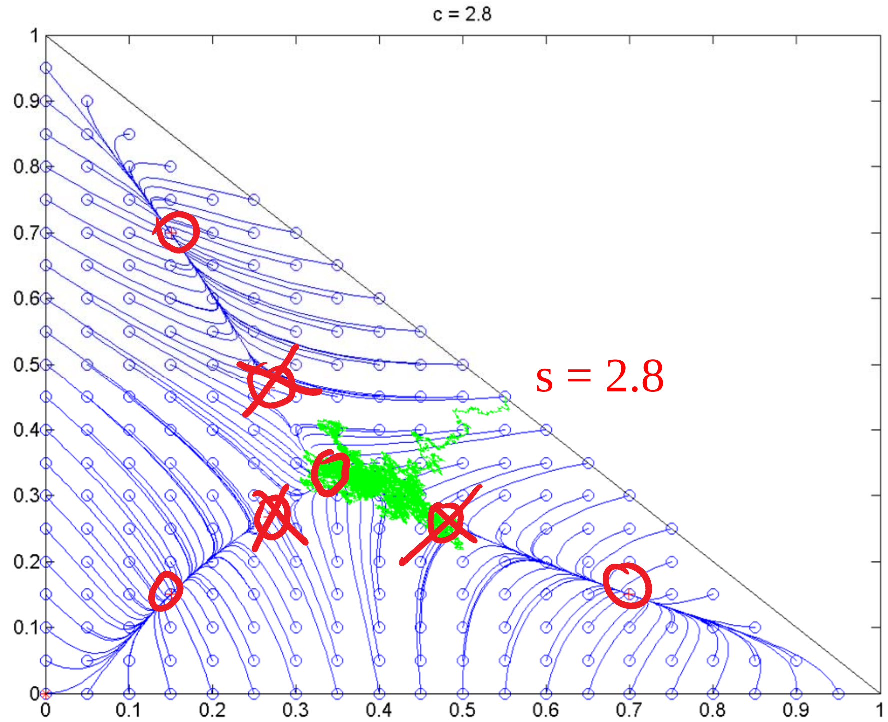
- We obtain
$$\frac{dx_1}{dt} = -x_1 e^{-sx_1} + \frac{1}{2} x_2 e^{-sx_2} + \frac{1}{2} x_3 e^{-sx_3}$$
$$\frac{dx_1}{dt} = -x_2 e^{-sx_2} + \frac{1}{2} x_1 e^{-sx_1} + \frac{1}{2} x_3 e^{-sx_3}$$

with $x_3 = 1 - x_1 - x_2$

Proba	Delta to (x_1, x_2)
$x_1\left(1-\frac{s}{N}\right)^{Nx_1-1} \times \frac{1}{2}$	$\frac{1}{N}(-1,+1)$
$x_1\left(1-\frac{s}{N}\right)^{Nx_1-1} \times \frac{1}{2}$	$\frac{1}{N}(-1,0)$
$x_2\left(1-\frac{s}{N}\right)^{Nx_2-1} \times \frac{1}{2}$	$\frac{1}{N}(+1,-1)$
$x_2\left(1-\frac{s}{N}\right)^{Nx_2-1} \times \frac{1}{2}$	$\frac{1}{N}(0,-1)$
$(1-x_1-x_2)\left(1-\frac{s}{N}\right)^{N(1-x_1-x_2)} \times \frac{1}{2}$	$\frac{1}{N}(+1,0)$
$(1-x_1-x_2)\left(1-\frac{s}{N}\right)^{N(1-x_1-x_2)} \times \frac{1}{2}$	$\frac{1}{N}(0,+1)$

The Fixed Point Method Applied to El Botellon

- $F(m^*) = 0$ has several solutions
- For $2.7456 \ldots < s < 3$ there are 7 fiexed points ?
- Which one should we take as approximation for the state probability when N is finite ?
- A possible answer : consider only stable points
  - This leaves 4 points

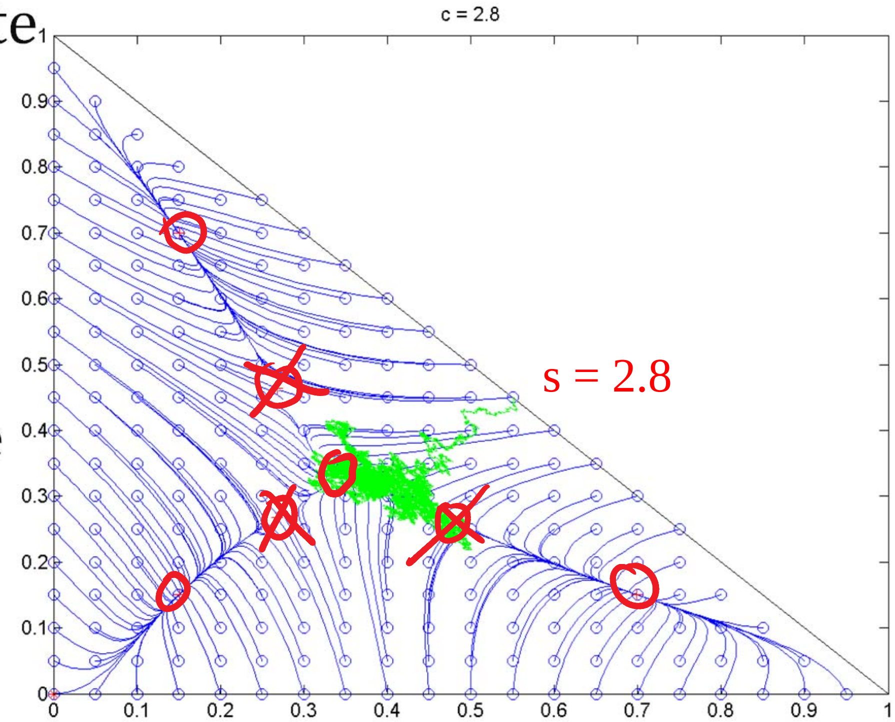


The Fixed Point Method may Provide Several Solutions

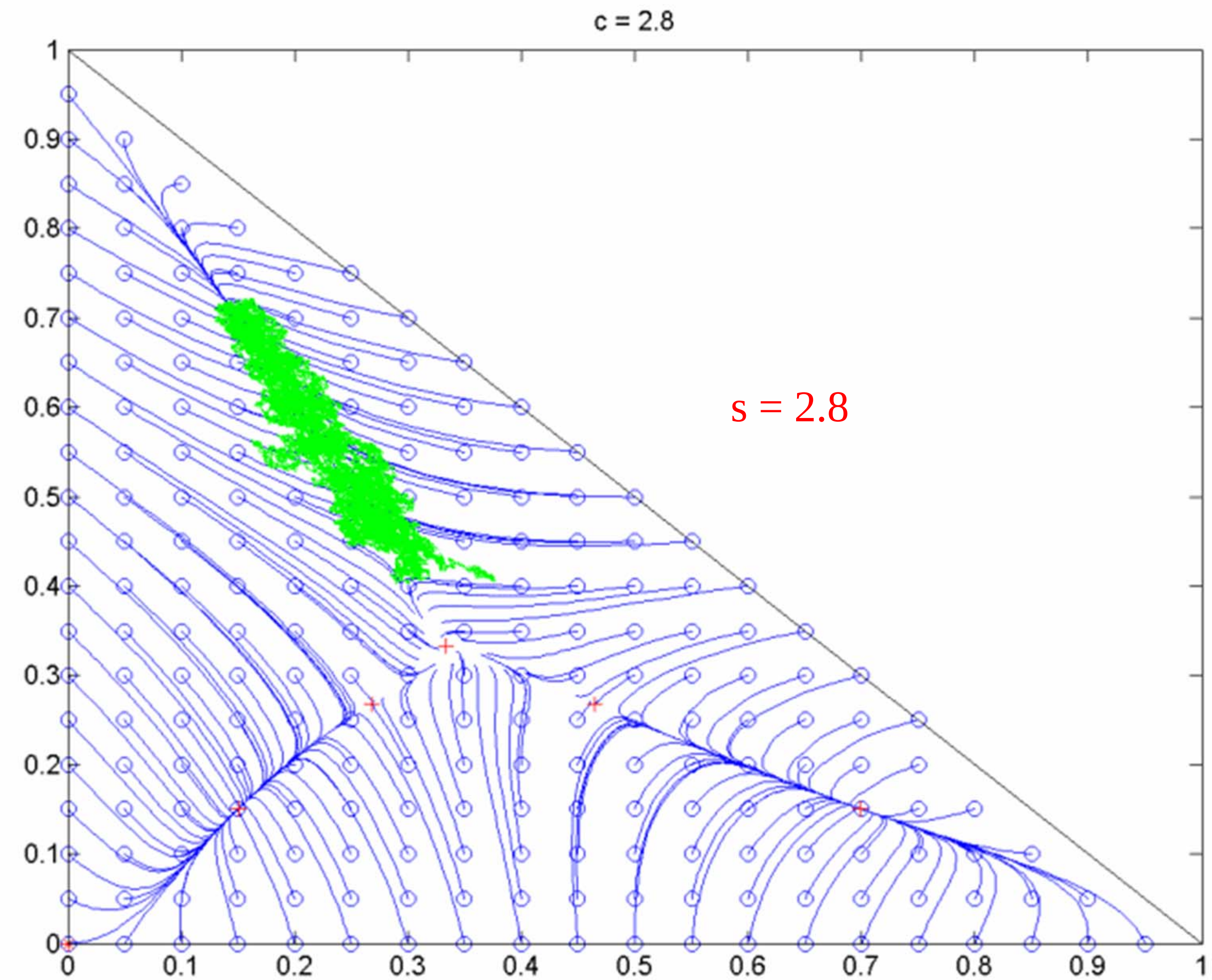
- The fixed point method finds the large N approximation of the state probabitiy for one object by solving $F(\vec{m}) = 0$

- This is the same as writing the balance equation and making the decoupling assumption in stationary regime

- There is an apparent contradiction in the method

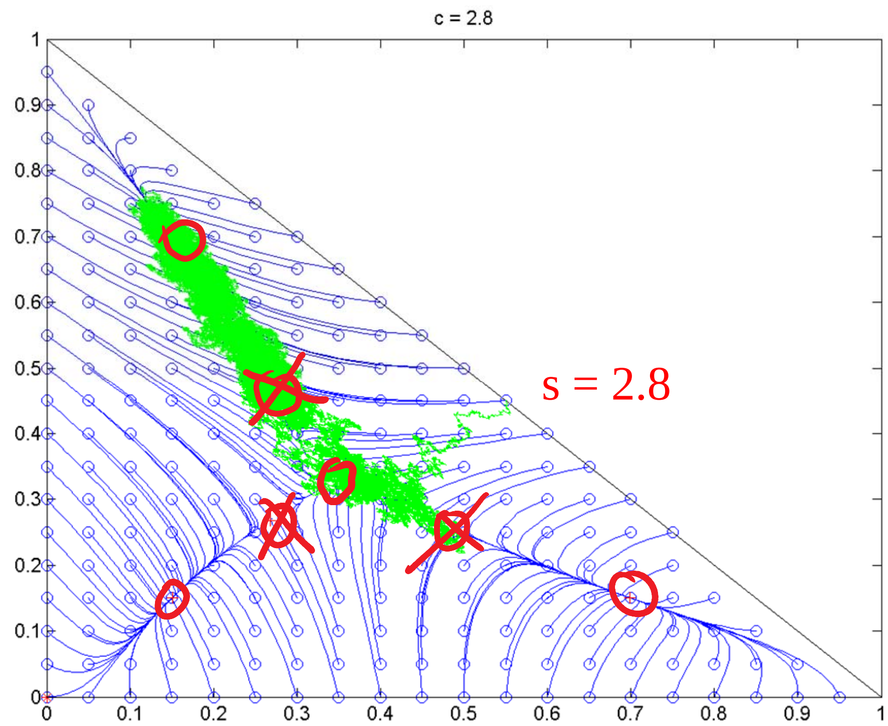


Simulations for large *t*



Simulations for large *t*

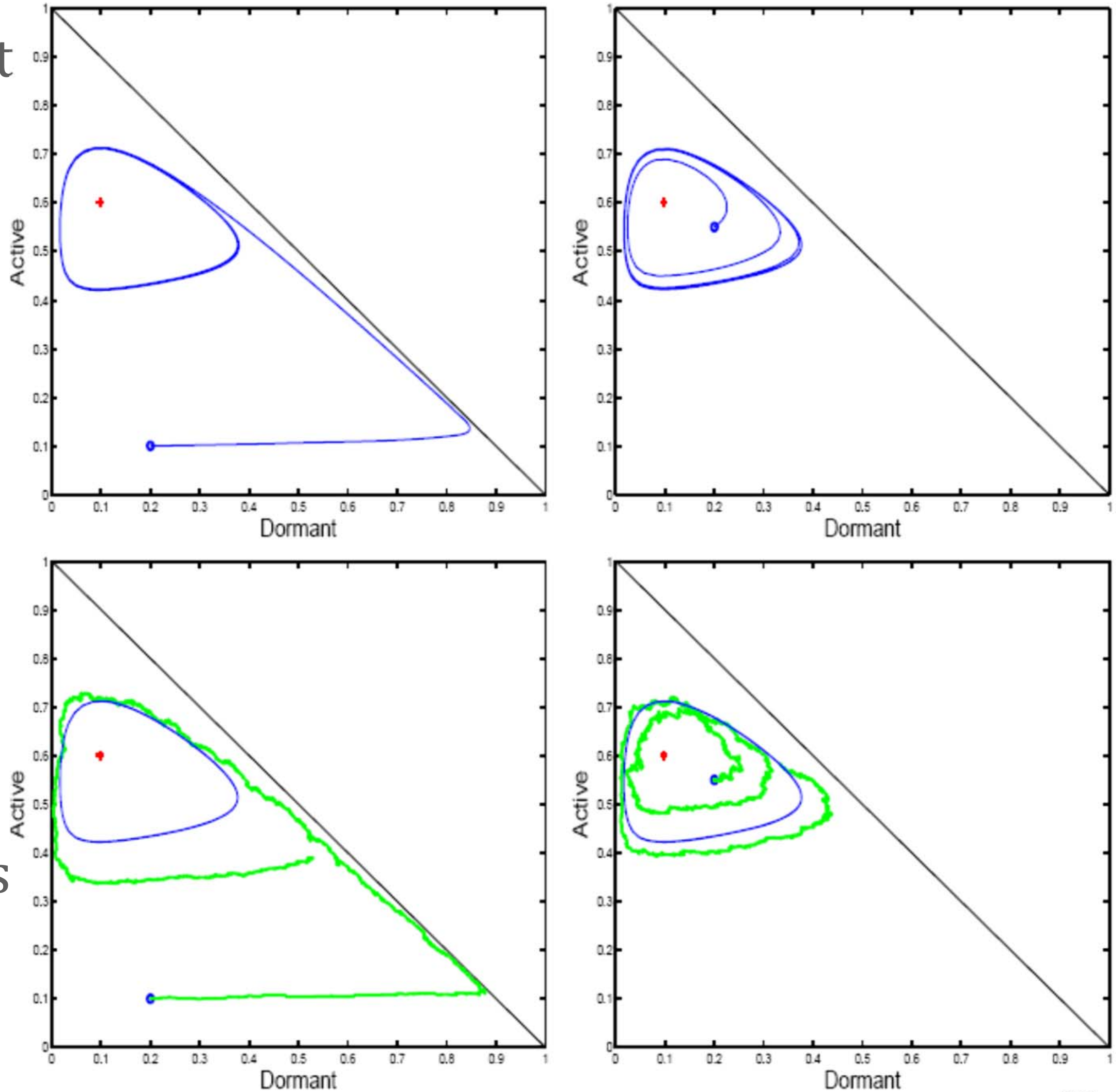
- If we wait long enough, the simulation jumps from one stable fixed point to another one



A Case with a Unique Fixed Point

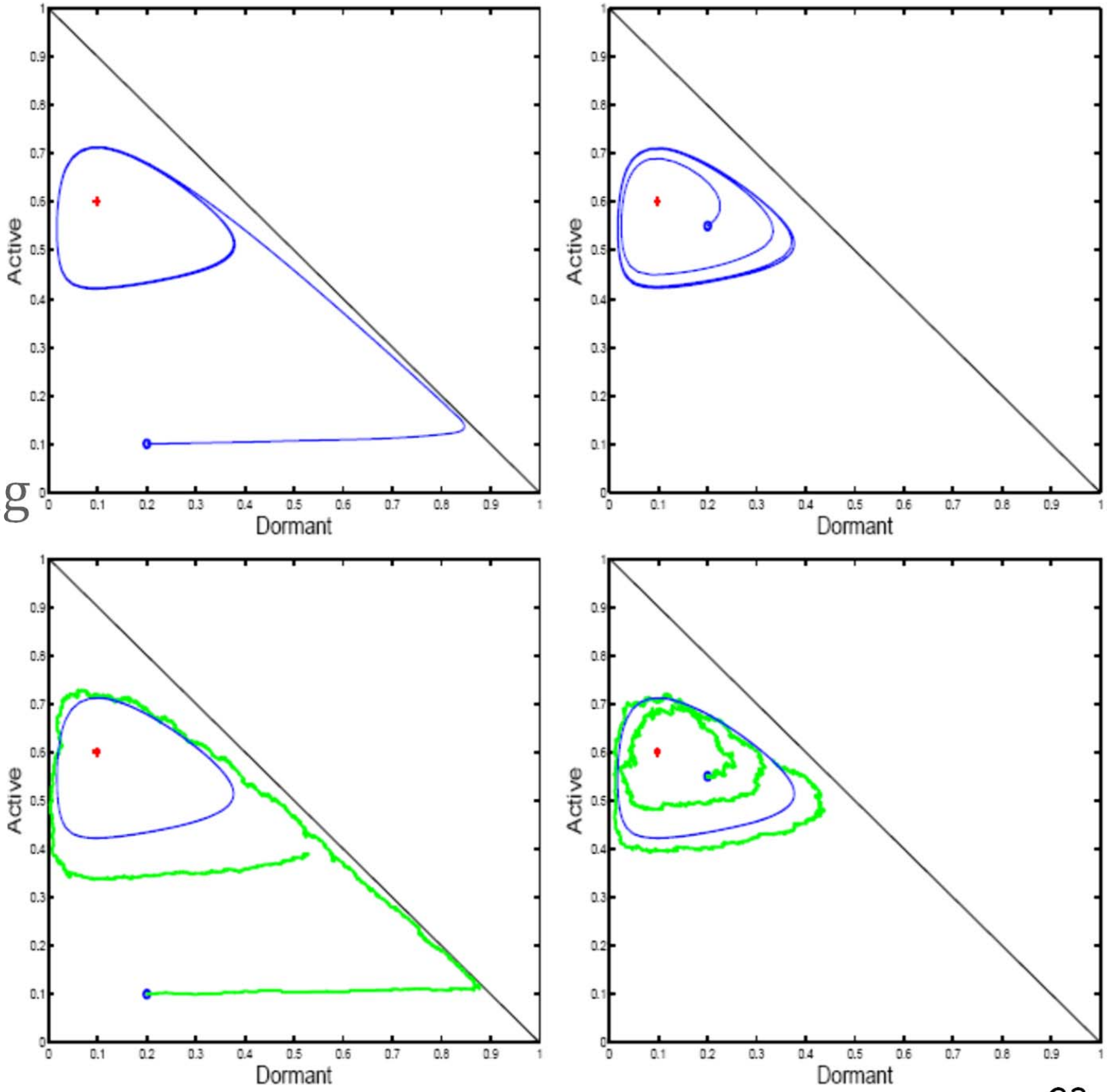
- Same as before except for one parameter value :
  $h = 0.1$ instead of 0.3

- The ODE does not converge to a unique attractor (limit cycle)
- The equation
  $$F(\vec{m}) = 0$$
  has a unique solution (red cross) but it does not give a good approximation of the simulation



The Fixed Point Method is Incorrect in This Case

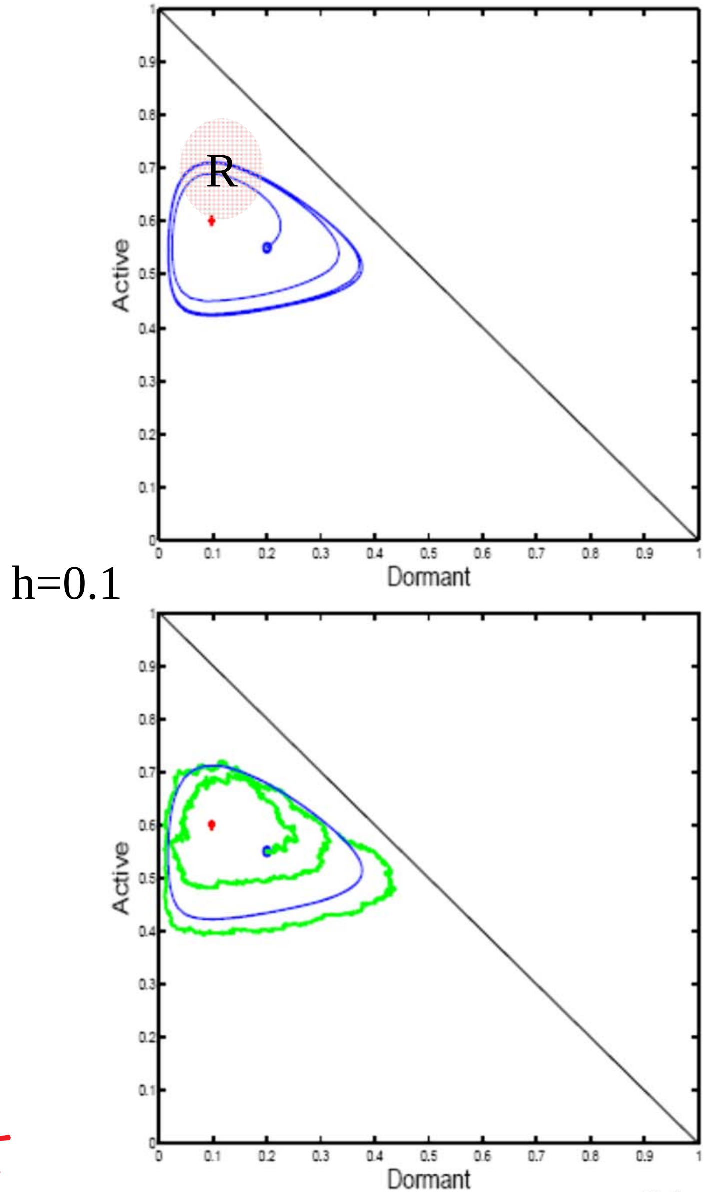
- The equation
  $$F(\vec{m}) = 0$$
  has a unique solution (red cross)
- However, there *is* convergence to mean field, hence decoupling assumption should hold.
- Where is the catch ?



Here Decoupling Assumption Does not Hold in Stationary Regime

- In stationary regime, $\vec{m}(t) = \big(D(t), A(t), S(t)\big)$ follows the limit cycle
- Assume you are in stationary regime (simulation has run for a long time) and you observe that one node, say $n = 1$, is in state ‘A’
- It is more likely that $m(t)$ is in region R
- Therefore, it is more likely that some other node, say $n = 2$, is also in state ‘A’

- Nodes are not independent – they are *synchronized*



Where is the Catch ?

Markov chain is ergodic

$$\mathbb{P}(X_1^N(t/N) = i \text{ and } X_1^N(t/N) = j) \xrightarrow{t \to \infty} \pi_{i,j}^N$$

Mean field

$$\downarrow N \to \infty \qquad\qquad \downarrow N \to \infty$$

$$\mu_i(t)\mu_j(t) \quad \xrightarrow{?} \quad ???$$

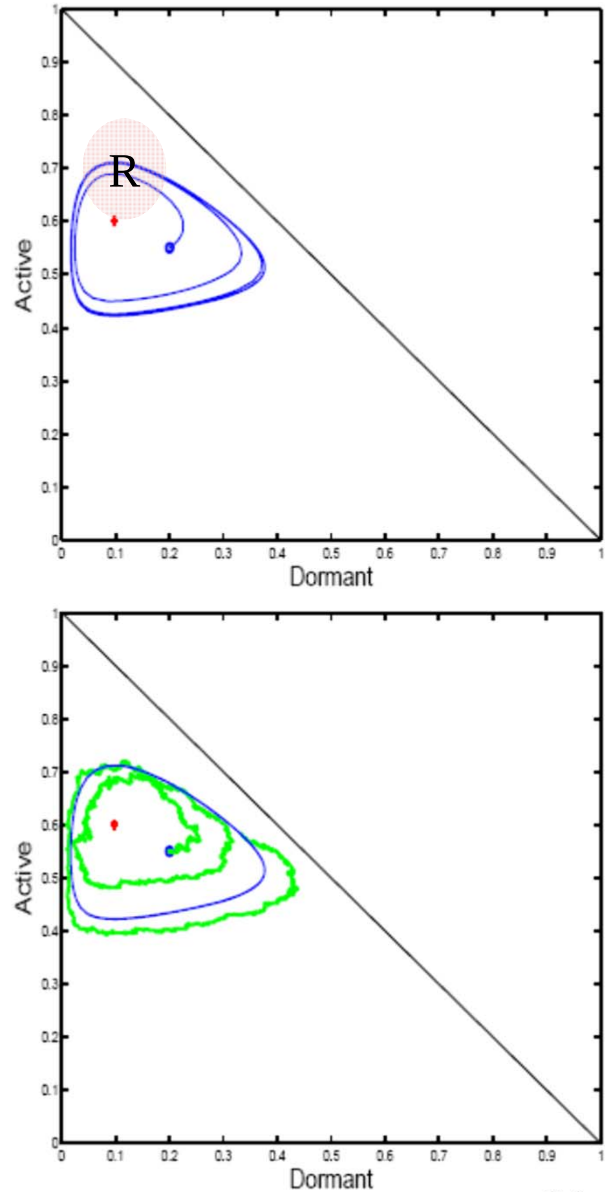
- The *decoupling assumption may not hold in stationary regime*, even for perfectly regular models

(exchange of limits may not hold)

The mean field property in stationary regime

- A correct statement is:

Conditional to the value of the mean field limit $m(t)$, 2 arbitrary nodes are asymptotically and independent and distributed like $m(t)$



Example: 802.11 Analysis, Bianchi's Formula

802.11 single cell

m_i = proba one node is in backoff stage I

β= attempt rate

γ = collision proba

See [Benaim 2008] for this analysis

ODE for mean field limit

$$\frac{dm_0}{d\tau} = -m_0 q_0 + \beta(\vec{m})\left(1 - \gamma(\vec{m})\right) + q_K m_K \gamma(\vec{m})$$

$$\frac{dm_i}{d\tau} = -m_i q_i + m_{i-1} q_{i-1} \gamma(\vec{m}) \qquad i = 1, \ldots, K$$

$$\beta(\vec{m}) = \sum_{i=0}^{K} q_i m_i$$

$$\gamma(\vec{m}) = 1 - e^{-\beta(\vec{m})}$$

Solve for Fixed Point:

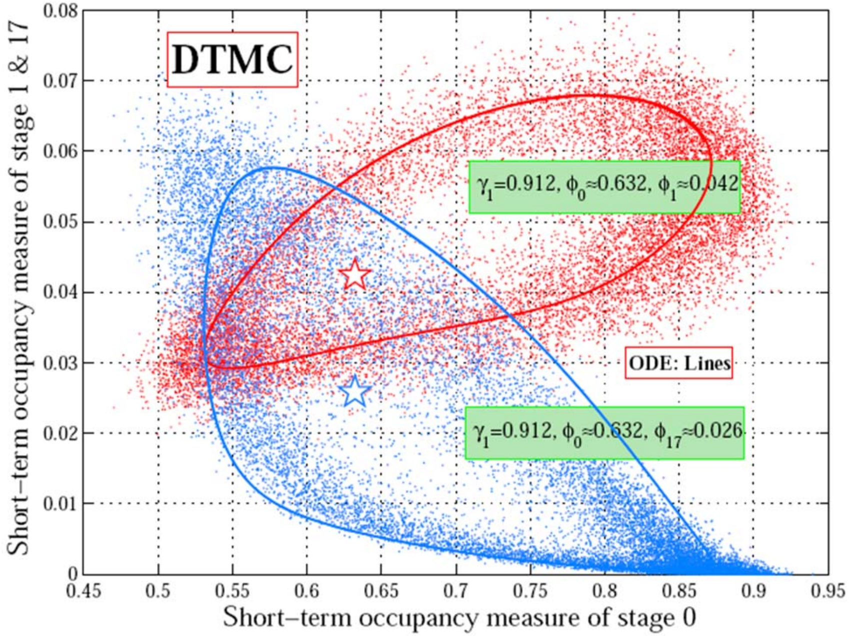
$$m_i = \frac{\gamma^i}{q_i} \frac{1}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

Bianchi's Fixed Point Equation [Bianchi 1998]

$$\gamma = 1 - e^{-\beta}$$

$$\beta = \frac{\sum_{k=0}^{K} \gamma^k}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

Example: 802.11 with Heterogeneous Nodes



- [Cho 2012]

  Two classes of nodes with heterogeneous parameters (restransmission probability)

  Fixed point equation has a unique solution, but this is not the stationary proba

  There is a limit cycle

Checkpoint

- The fixed point method seems reasonable to study the system in stationary regime
- But it may not give the correct answer, even if there is a unique fixed point
- We can say a bit more

3

ASYMPTOTIC RESULTS

A Generic Result For Stationary Regime

- Original system (stochastic):
  - $(X^N(t))$ is Markov, finite, discrete time
  - Assume it is irreducible, thus has a unique stationary proba ν^N
  - Let ϖ^N be the corresponding stationary distribution for $M^N(t)$, i.e.

    $P\left(M^N(t)=(x_1,\ldots,x_I)\right) = \varpi^N(x_1,\ldots,x_I)$ for x_i of the form k/n, k integer
- Theorem [e.g. Benaim 2008]

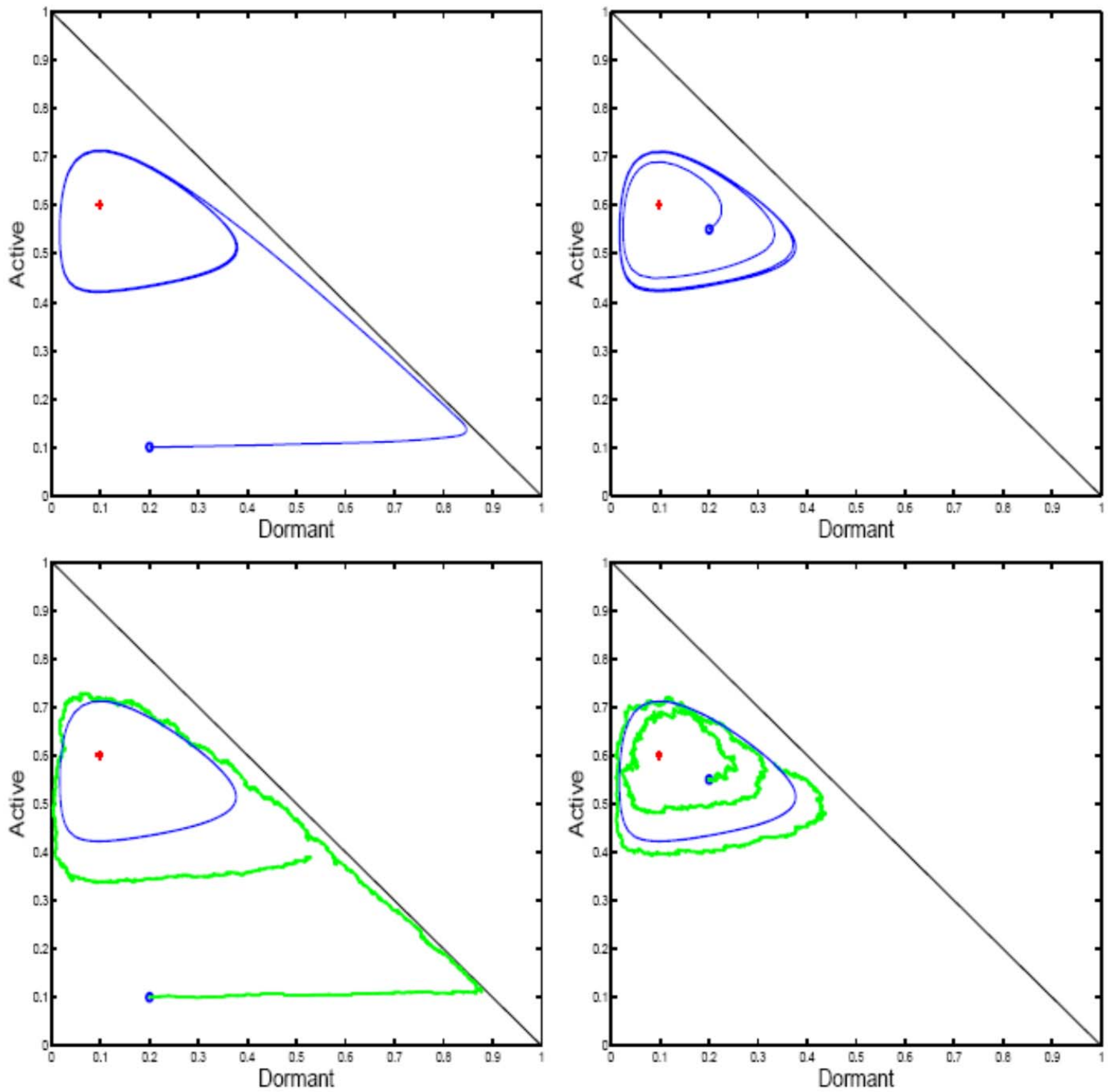
Theorem 3 *The support of any limit point of ϖ^N is a compact set included in the Birkhoff center of Φ.*

Birkhoff Center: closure of set of points s.t. $m \in \omega(m)$
Omega limit: $\omega(m)$ = set of limit points of orbit starting at m

- Here:
  Birkhoff center = limit cycle ∪ fixed point

- The theorem says that the stochastic system for large N is close to the Birkhoff center,

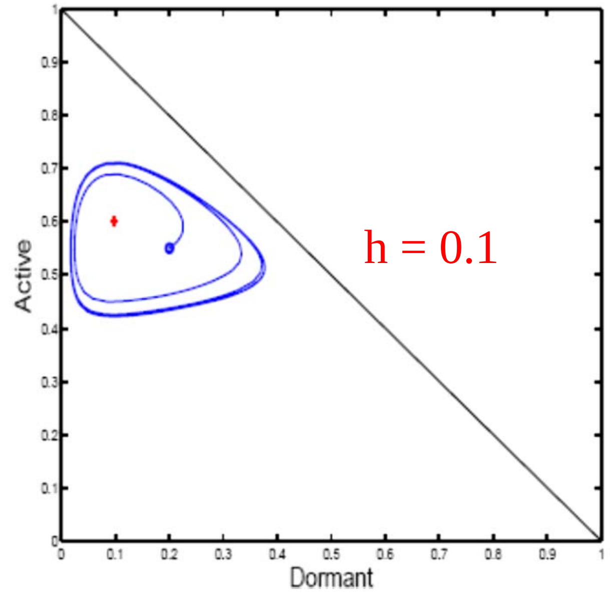
  i.e. the stationary regime of ODE is a good approximation of the stationary regime of stochastic system

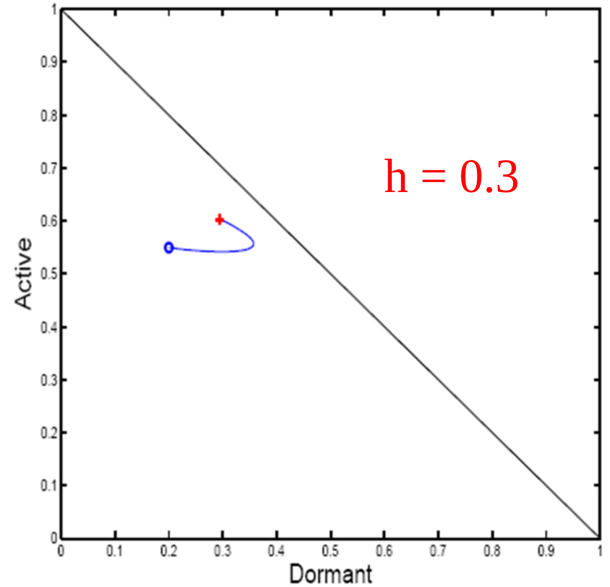


Take Home Message

- The stationary behaviour of the mean field limit is a good approximation of the original system

- But… the stationary behaviour of a deterministic system (i.e. an ODE $\frac{dm}{dt} = F(m)$) is not always obtained by looking for fixed points (i.e. $F(m) = 0$) !

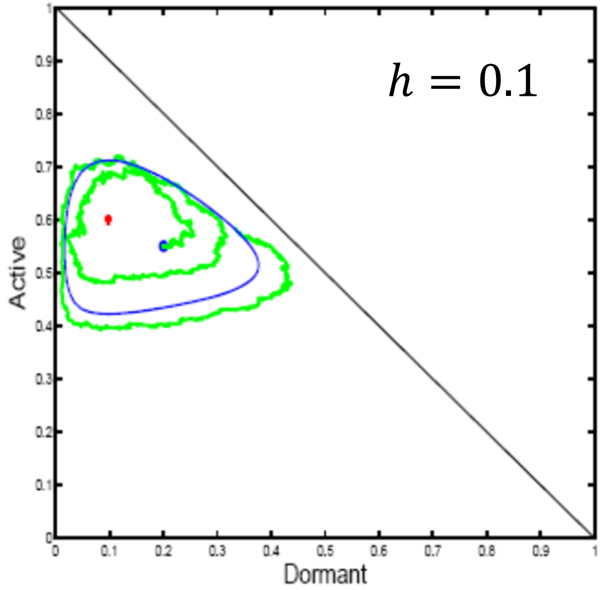




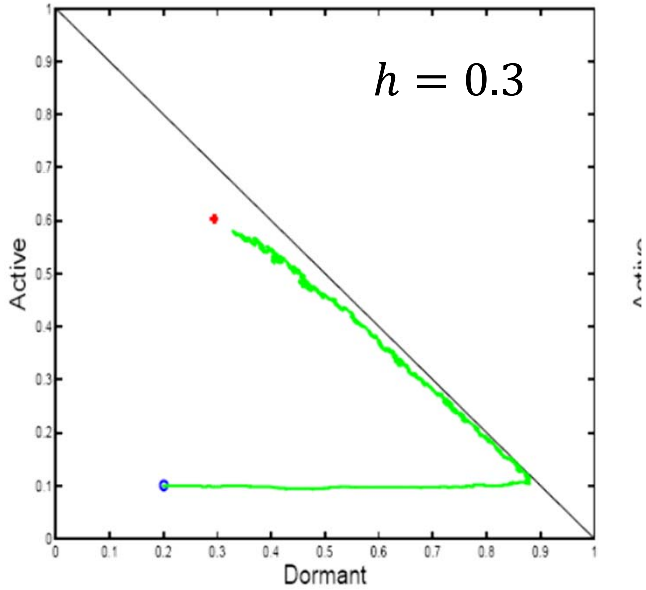
The Good Case

- (H) ODE has a unique fixed point to which all trajectories converge
- Theorem [Benaim 2008] : If (H) is true then the limit of stationary distribution of M^N is concentrated on this fixed point
  - i.e., under (H), the fixed point method and the decoupling assumptions are justified

Fixed point method
does not work



Fixed point method
works



In the Reversible Case, the Fixed Point Method Always Works

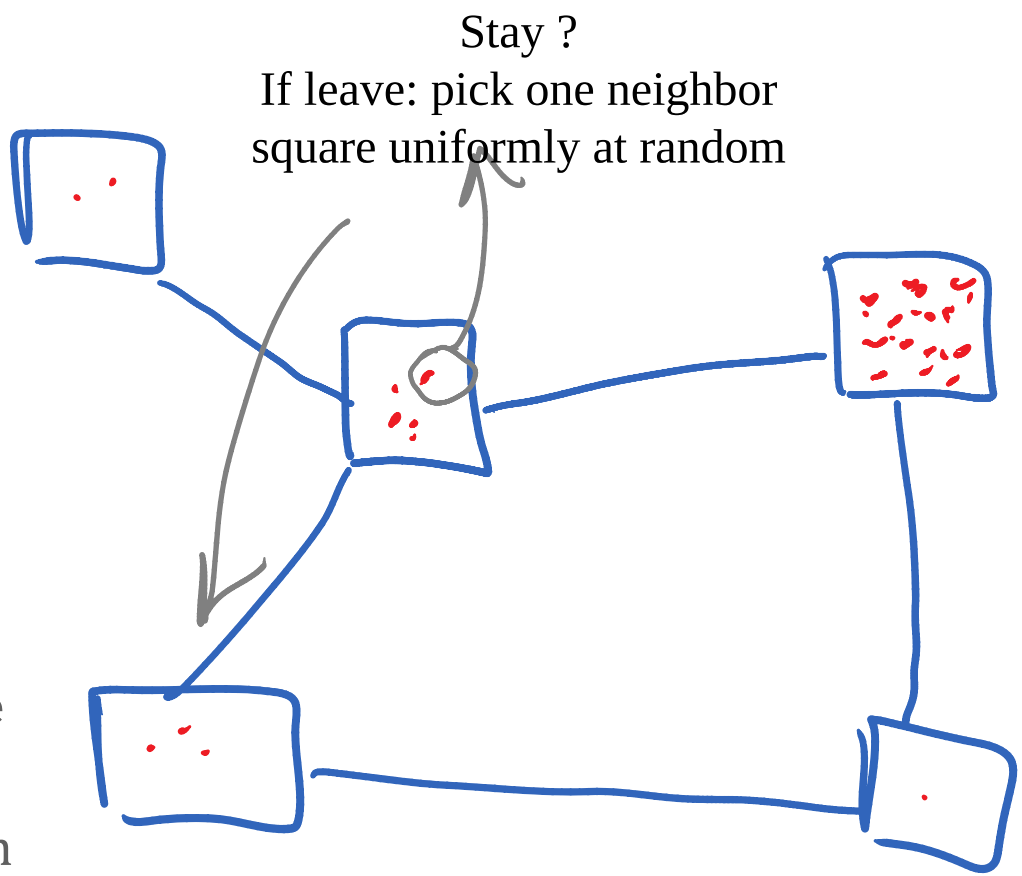
- **Definition** Markov Process $X(t)$ with transition rates $q(i,j)$ is reversible iff
  1. it is ergodic 2. $p(i)\, q(i,j) = p(j)\, q(j,i)$ for some p

- If process with finite N is reversible, the stationary behaviour is determined only by fixed points of the mean field limit [Le Boudec 2010]

Example of Reversible Case: El Botellon

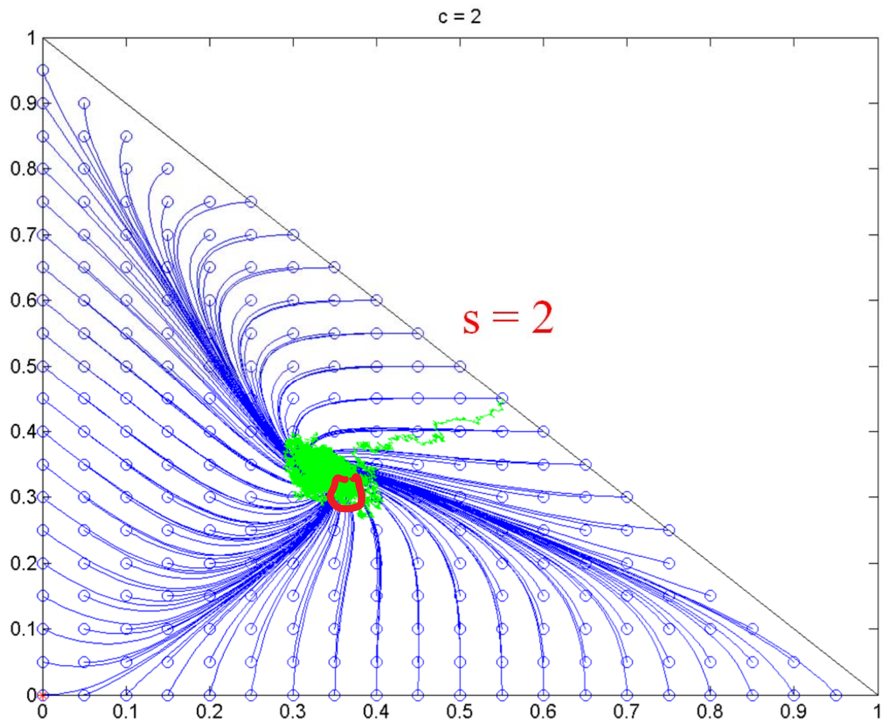
- The Markov process with finite N is reversible

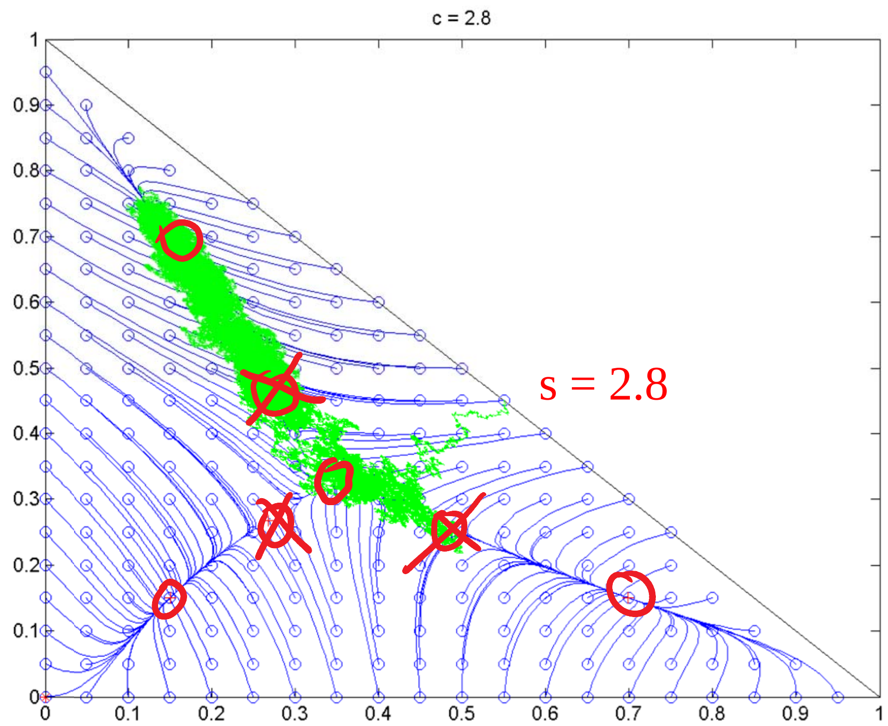
- Follows from the theory of product-form queuing network [Le Boudec 2012]
  - It is a product-form queuing network
  - Routing process is reversible
  - A product-form queing network is reversible as soon as the routing process is reversible (otherwise not)



All trajectories of ODE converge to the unique fixed point
Fixed point method is valid
The proba that a person is in square i is $\approx \frac{1}{3}$

All trajectories of ODE has four stable fixed points
The occupancy measure is concentrated around the four stabel fixed points (metastability)





4

HOW TO USE MEAN FIELD IN STATIONARY REGIME

A Correct Method in Order to Use the Mean Field Approximation in Stationary Regime

- 1. Write dynamical system equations *in transient regime*

- 2. Study the *stationary regime of* dynamical system
  - **if** converges to unique stationary point m^* **then** make fixed point assumption
  - **else** objects are coupled in stationary regime by mean field limit $m(t)$

- Hard to predict outcome of 2 (except for reversible case)

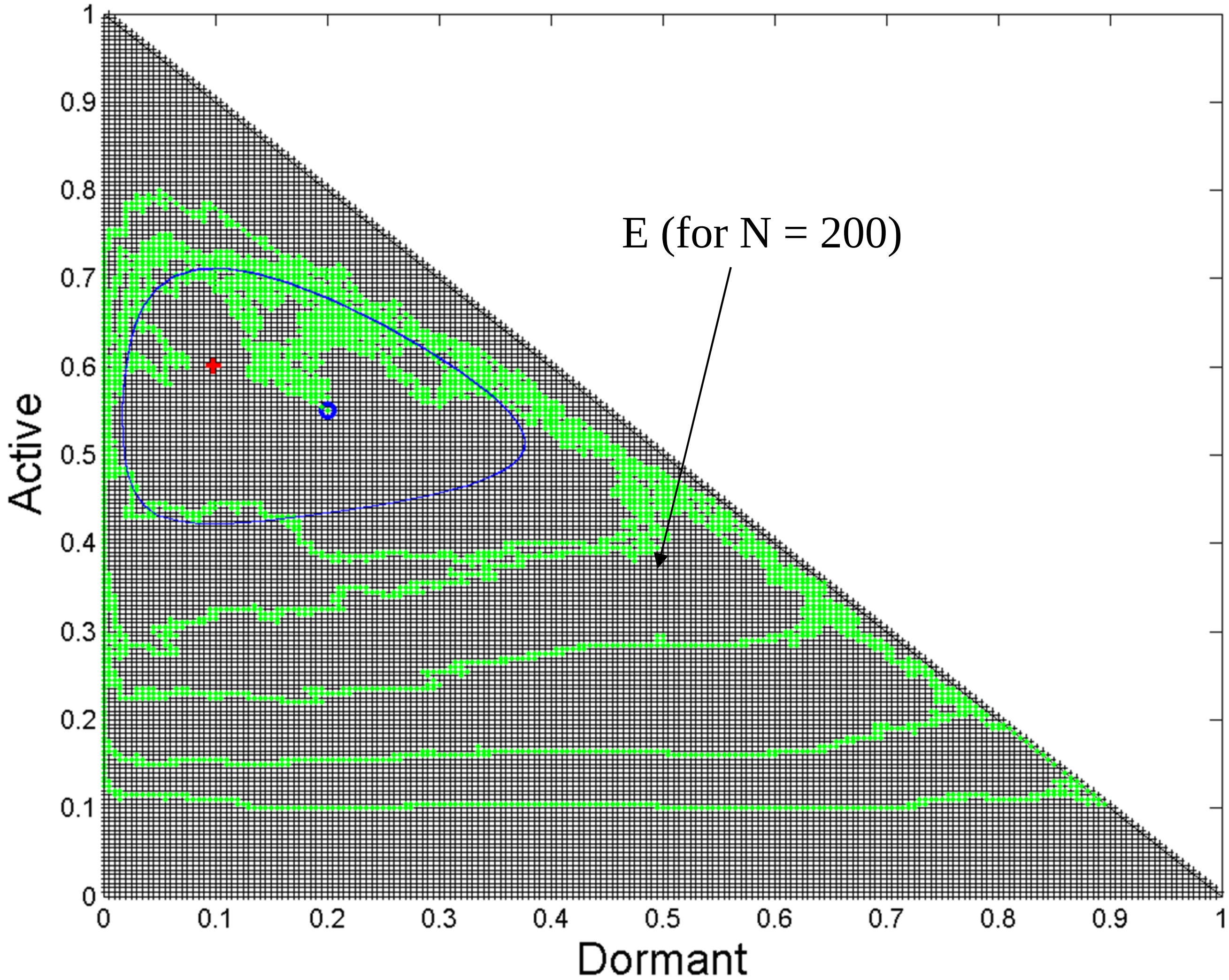
Quiz : $M^N(t)$ is a Markov chain on $E = \{(a,b,c) \geq 0,\ \ a + b + c = 1,\ a,b,c \text{ multiples of } 1/N\}$

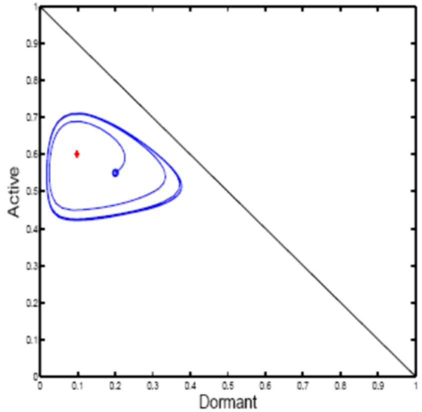
A. $M^N(t)$ is periodic, this is why there is a limit cycle for large N.

B. For large N, the stationary proba of M^N tends to be concentrated on the blue cycle.

C. For large N, the stationary proba of M^N tends to a Dirac.

D. $M^N(t)$ is not ergodic, this is why there is a limit cycle for large N.





Randomness May Come Back in the Mean Field Limit

for $h = 0.1$:

$$P\left(X_1^N\left(\frac{t}{N}\right) = i \text{ and } X_2^N\left(\frac{t}{N}\right) = j\right) \approx \frac{1}{T}\int_0^T m_i(t)m_j(t)dt$$

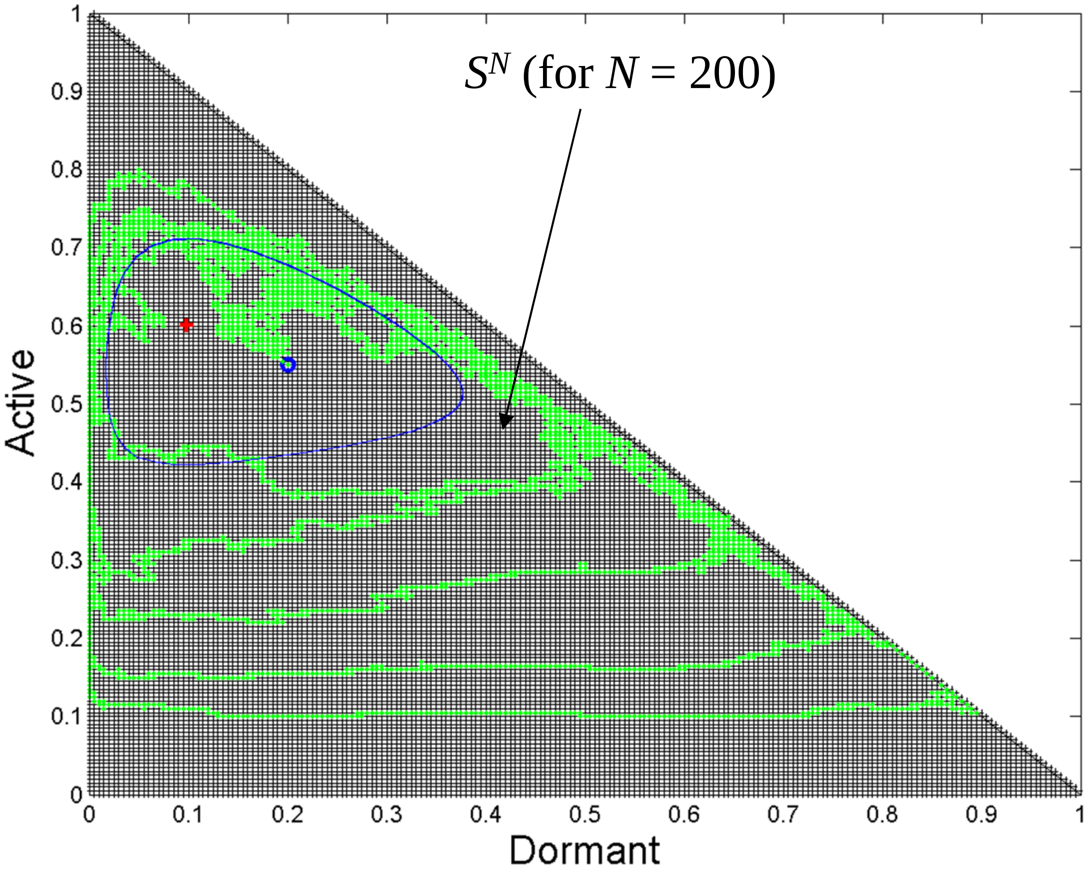
$$\neq \left(\int_0^T m_i(t)\,dt\right)\left(\int_0^T m_j(t)\,dt\right)$$

where T is the period of the limit cycle

The mean field limit $m(t)$ is random in the stationary regime, even if the mean field process is deterministic

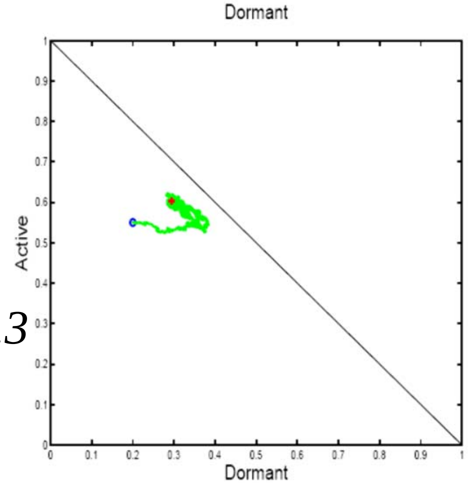
Stationary Behaviour of Mean Field Limit is not predicted by Structure of Markov Chain

- $M^N(t)$ is a Markov chain on $S^N=\{(a, b, c) \geq 0, a + b + c = 1,\ a, b, c \text{ multiples of } 1/N\}$
- $M^N(t)$ is ergodic and aperiodic, for any value of h

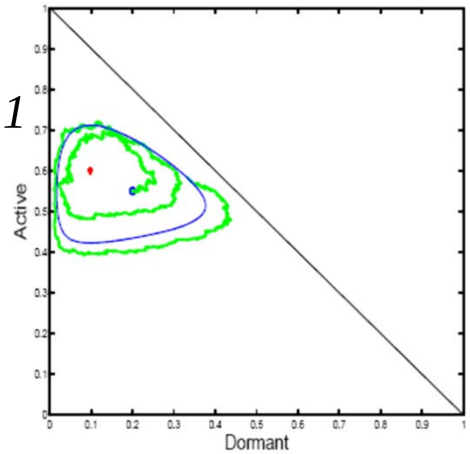


- Depending on h, there is or is not a limit cycle for $m(t)$

$h = 0.3$



$h = 0.1$



Conclusion

- Mean field models are frequent in large scale systems
- Validity of approach is often simple by inspection
- Mean field is both
  - ODE for fluid limit
  - Fast simulation using decoupling assumption
- Decoupling assumption holds at finite horizon; may not hold in stationary regime (except for reversible case)
- Study the stationary regime of the ODE !

  (instead of computing the stationary proba of the Markov chain)

Questions ?

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