# Mean Field Methods for Computer and Communication Systems

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# 1

# MEAN FIELD INTERACTION MODEL

# Mean Field

- A *model* introduced in Physics
  - interaction between *particles* is via distribution of states of all particle
- An *approximation* method for a large collection of particles
  - assumes *independence* in the master equation
- Why do we care in information and communication systems ?
  - Model interaction of many objects:
  - Distributed systems, communication protocols, game theory, self-organized systems

# A Few Examples Where Applied


[1] L. Afanassieva, S. Popov, and G. Fayolle. Models for transporation networks. *Journal of Mathematical Sciences, 1997 – Springer*.

[2] F. Baccelli, A. Chaintreau, D. De Vleeschauwer, and D. R. McDonald. Http turbulence, May 2004.

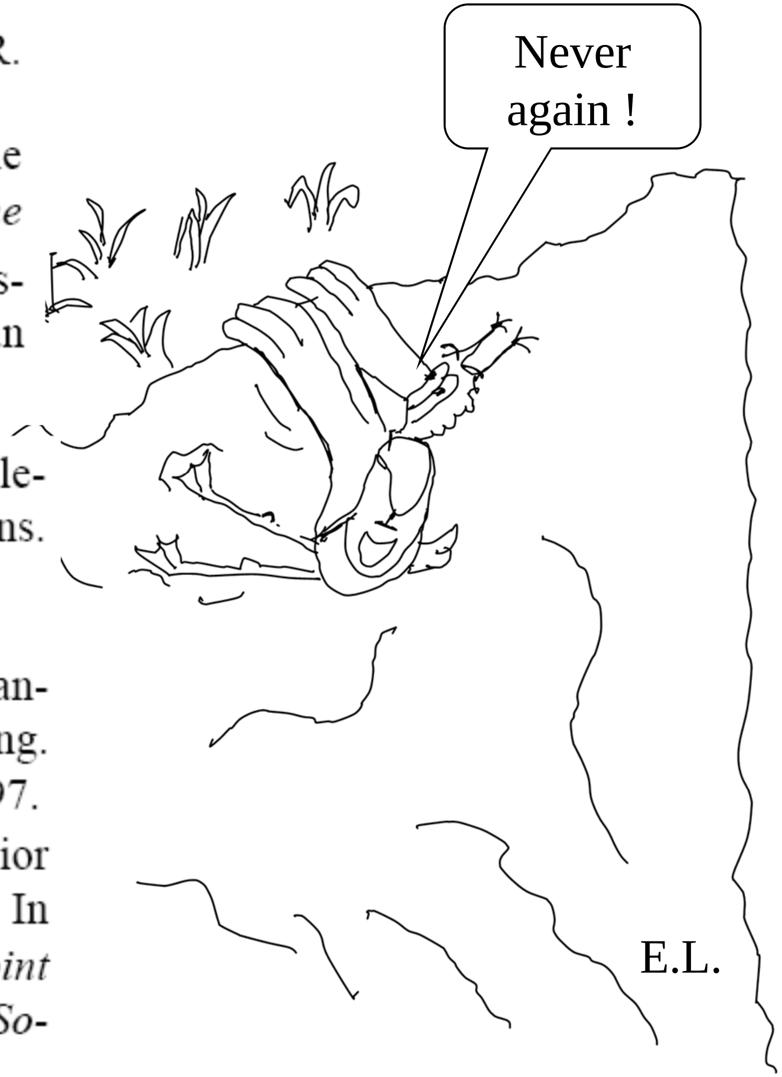
[3] F. Baccelli, M. Lelarge, and D. McDonald. Metastable regimes for multiplexed tcp flows. In *Proceedings of the*

[5] M.-D. Bordenave, Charles and A. Proutiere. A particle system in interaction with a rapidly varying environment: Mean field limits and applications. arXiv:math/0701363v2.

[11] S. Kumar and L. Massoulié. Integrating streaming and file-transfer internet traffic: Fluid and diffusion approximations. MSR-TR-2005-160.

[16] Y. M. Suhov and N. D. Vvedenskaya. Dobrushin's mean-field approximation for a queue with dynamic routing. *Markov Processes and Related Fields*, 3(4):493–526, 1997.

[17] P. Tinnakornsrisuphap and A. M. Makowski. Limit behavior of ecn/red gateways under a large number of tcp flows. In *Proceedings IEEE INFOCOM 2003, The 22nd Annual Joint Conference of the IEEE Computer and Communications Societies, San Franciso, CA, USA*, March 30 - April 3 2003.

# Mean Field Interaction Model

- Time is discrete (this talk) or continuous
- $N$ objects, $N$ large
- Object $n$ has state $X_n(t)$
- $(X^N{}_1(t), \ldots, X^N{}_N(t))$ is Markov
- Objects are observable only through their state
- "Occupancy measure" $M^N(t)$ = distribution of object states at time $t$

# Example: 2-Step Malware

- Mobile nodes are either
  - `S’ Susceptible
  - `D’ Dormant
  - `A’ Active
- Time is discrete
- Transitions affect 1 or 2 nodes

- State space is finite = {`S’ , `A’ ,`D’}

- Occupancy measure is M(t) = (S(t), D(t), A(t)) with S(t)+ D(t) + A(t) =1

S(t) = proportion of nodes in state `S’

[Benaïm and Le Boudec(2008)]

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
5. Recruitment by Dormant
   - S + D -> D + D

   Direct infection
   - S -> D
6. Direct infection
   - S -> A

# 2-Step Malware – Full Specification

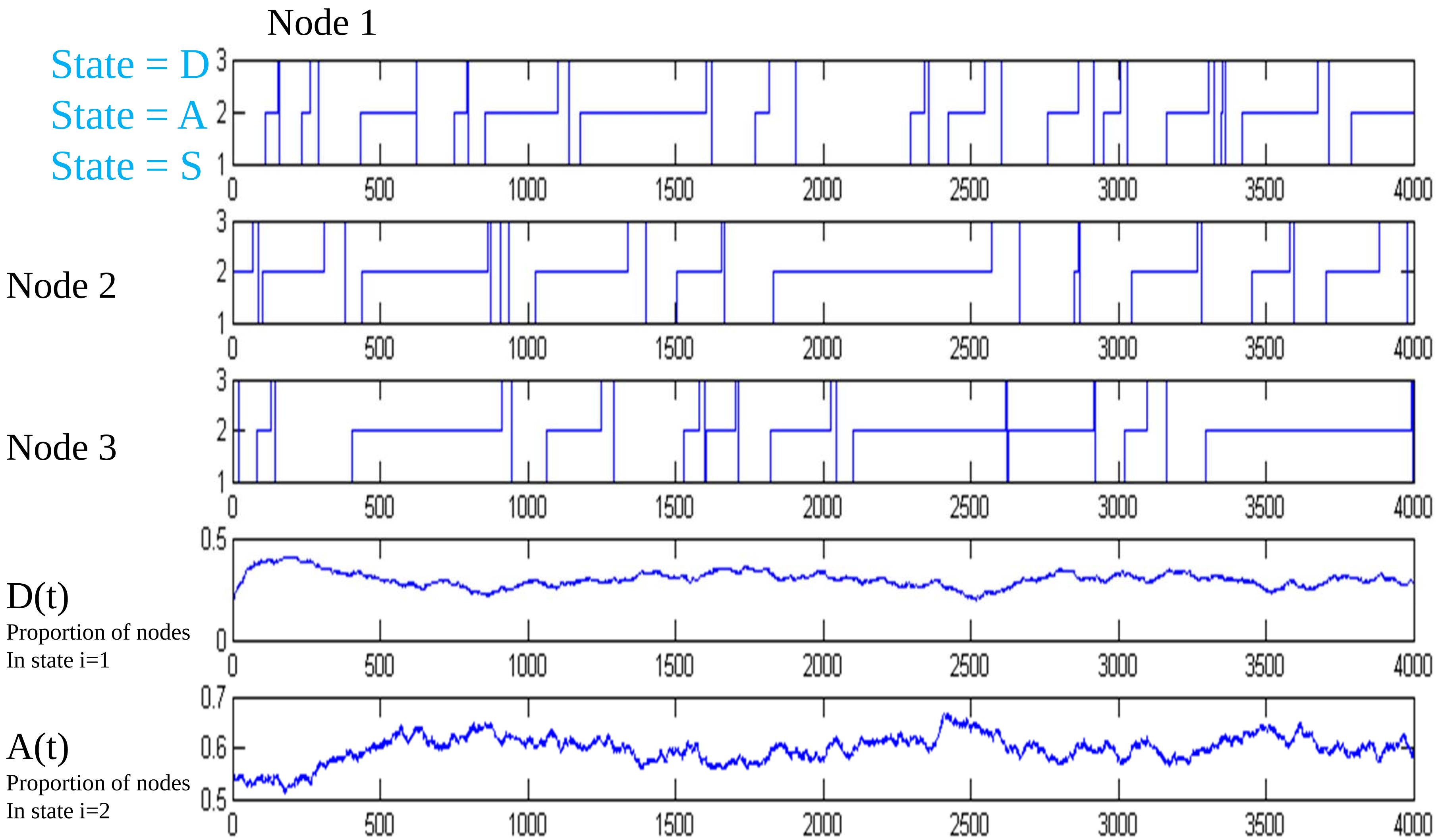
At every time step, pick one node unif at random

- If node is in state $D$:
  - With proba $\delta_D$ mutate to S
  - With proba $\lambda \frac{ND-1}{N}$, meet another $D$ node and both mutate to $A$
- If node is in state $A$:
  - With proba $\beta \frac{D}{h+D}$ change one $D$ node to $A$
  - With Proba $\delta_A$ mutate to $S$
- If node is in state $S$
  - With proba $rD$ meet a $D$ node and become infected $D$
  - With proba $\alpha_0$ become infected $D$
  - With proba $\alpha$ become infected $A$

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
5. Recruitment by Dormant
   - S + D -> D + D

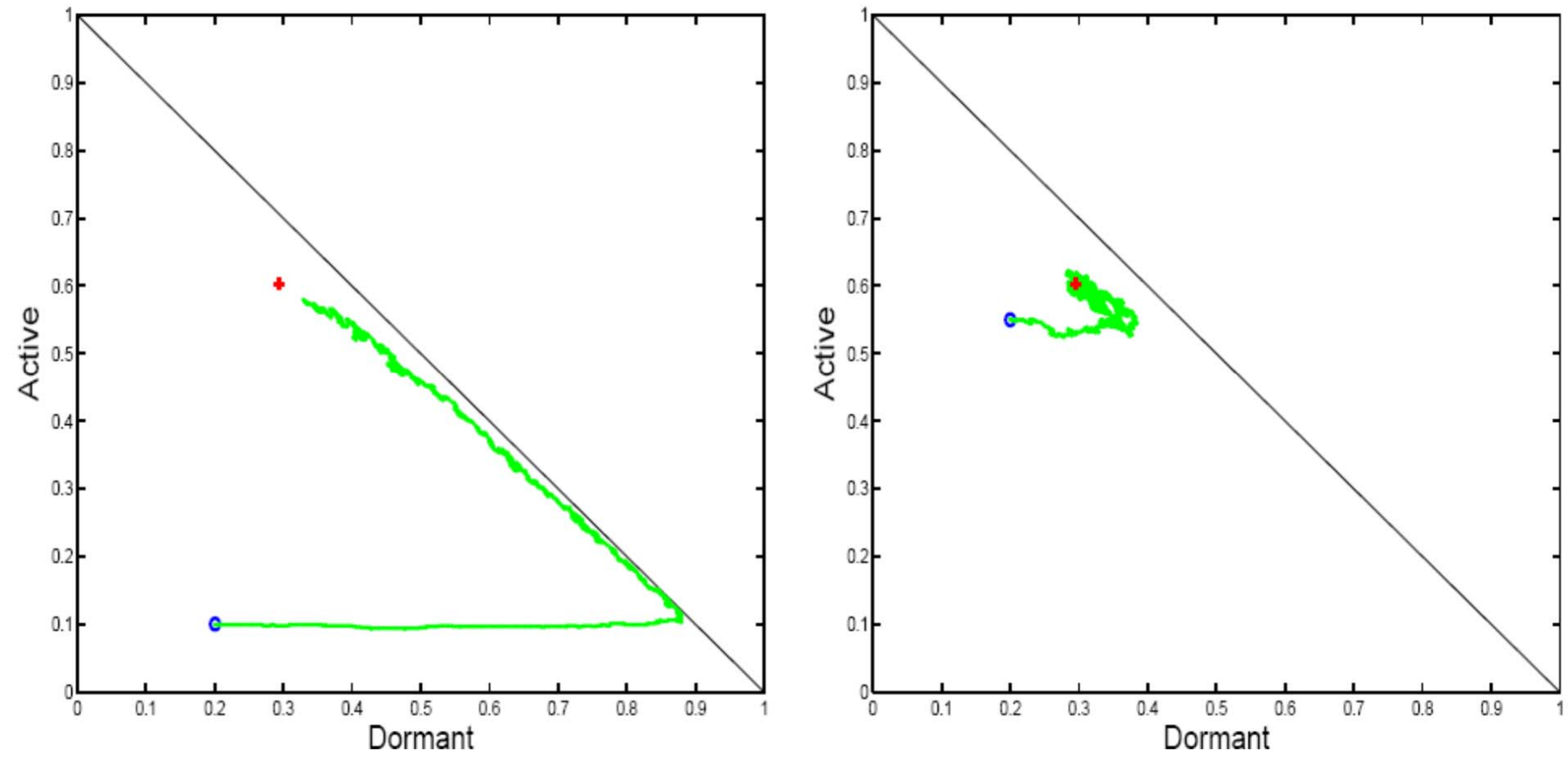
   Direct infection
   - S -> D
6. Direct infection
   - S -> A

# Simulation Runs, N=1000 nodes



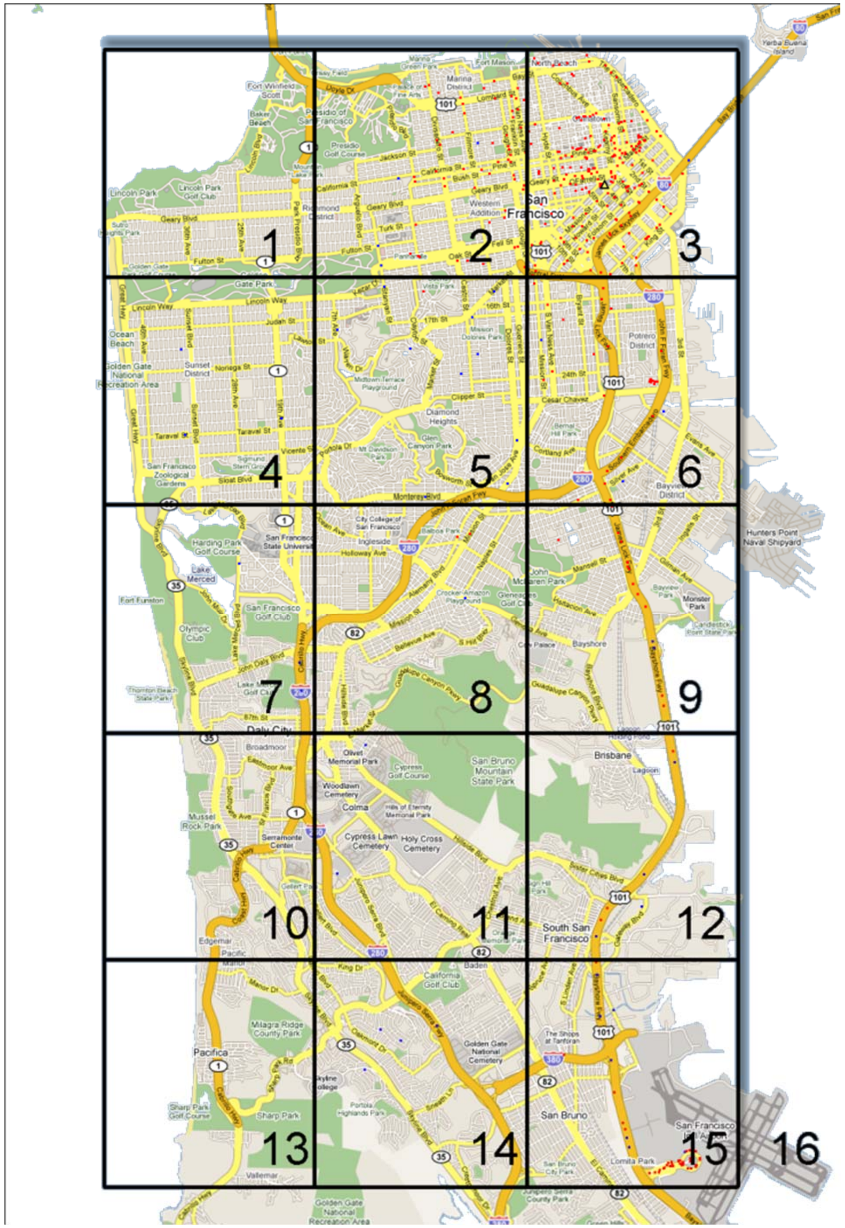

$$\beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001$$

# Sample Runs with N = 1000




$$\beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001$$

# The Importance of Being Spatial

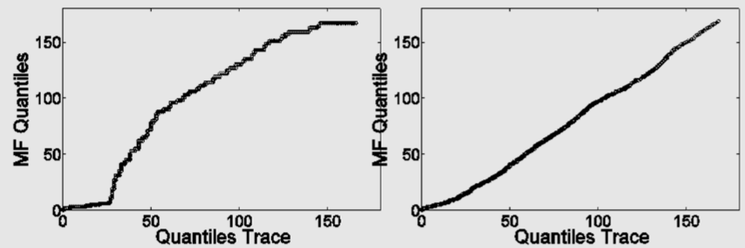



- Mobile node state = (c, t)
  c = 1 … 16 (position)
  t $\in R^+$ (age of gossip)

- Time is continuous
- Occupancy measure is
  $F_c(z,t)$ = proportion of nodes that at location $c$ and have age $\leq z$

  [Age of Gossip, Chaintreau et al.(2009)]

no class 16 classes

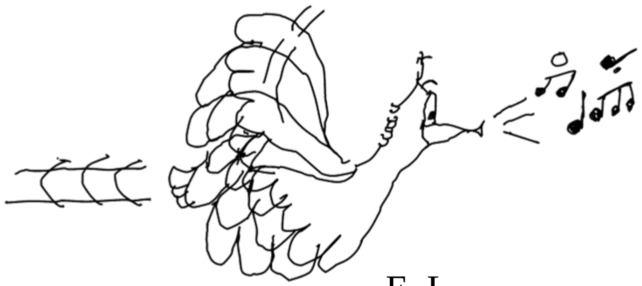



Qqplots simulation vs mean field

# What can we do with a Mean Field Interaction Model ?

- Large $N$ asymptotics, Finite Horizon
  - fluid limit of occupancy measure (ODE)
  - decoupling assumption (fast simulation)
- Issues
  - When valid
  - How to formulate the fluid limit

- Large $t$ asymptotic
  - Stationary approximation of occupancy measure
  - Decoupling assumption
- Issues
  - When valid



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# CONVERGENCE TO ODE

# To Obtain a Mean Field Limit we Must Make Assumptions about the Intensity *I(N)*

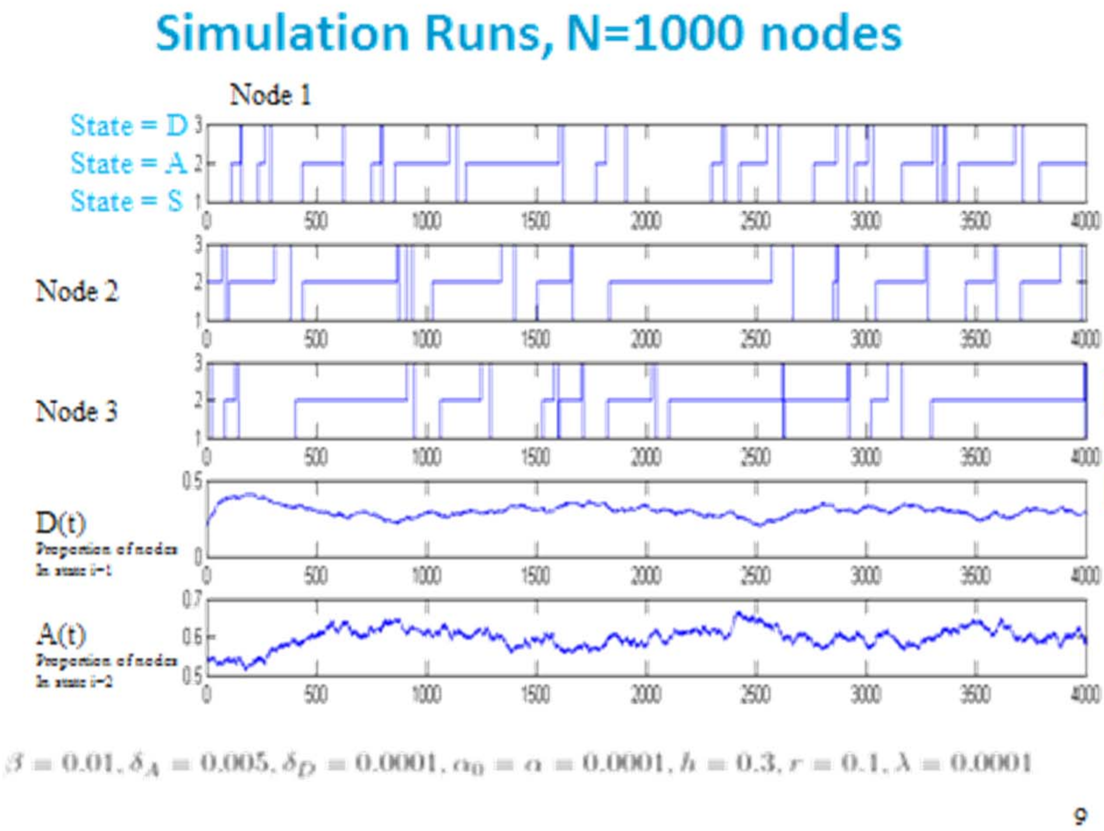
- $I(N)$ = (order of) expected number of transitions per object per time unit

- A mean field limit occurs when we re-scale time by $I(N)$ i.e. one time slot $\approx I(N)$
  i.e. we consider $X^N(t/I(N))$

- I(N) = O(1): mean field limit is in discrete time [Le Boudec et al (2007)]

  I(N) = O(1/N): mean field limit is in continuous time [Benaïm and Le Boudec (2008)] (this talk)

# Intensity for this model is $1/N$




- In one time step, the number of objects affected by a transition is 0, 1 or 2; mean number of affected objects is $O(1)$
- There are $N$ objects
- Expected number of transitions per time slot per object is $O\left(\frac{1}{N}\right)$

# The Mean Field Limit

- Under very general conditions (given later) the occupancy measure converges, in law, to a deterministic process, *m(t),* called the *mean field limit*

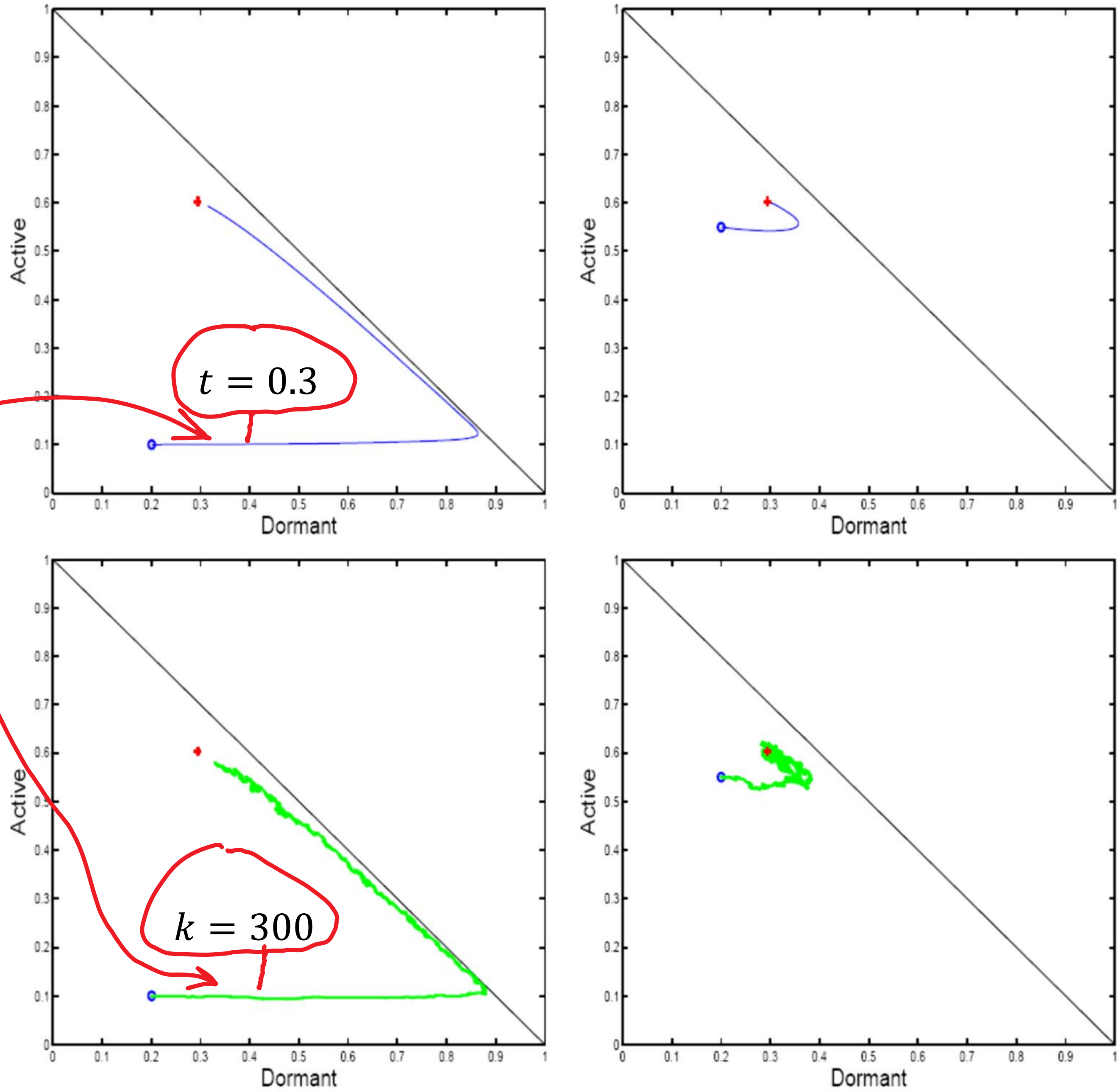
$$M^N\left(\frac{t}{I(N)}\right) \to m(t)$$

- Finite State Space => ODE

**Mean Field Limit**
**N = +∞**

$$m(t) \approx M^N(t\,N)$$

**Stochastic system**
**N = 1000**



# Sufficient Conditions for Convergence

- [Kurtz 1970], see also [Bordenav et al 2008], [Graham 2000]
- Sufficient condition verifiable by inspection:
  - probabilities at every time slot have a limit when $N \to \infty$
  - [Benaïm and Le Boudec(2008), Ioannidis and Marbach(2009)]
    - Let $W^N(k)$ be the number of objects that do a transition in time slot $k$. Note that $\mathbb{E}\left(W^N(k)\right) = NI(N)$, where $I(N) \stackrel{\text{def}}{=}$ intensity. Assume

$$\mathbb{E}\left(W^N(k)^2\right) \leq \beta(N) \quad \text{with} \quad \lim_{N\to\infty} I(N)\beta(N) = 0$$

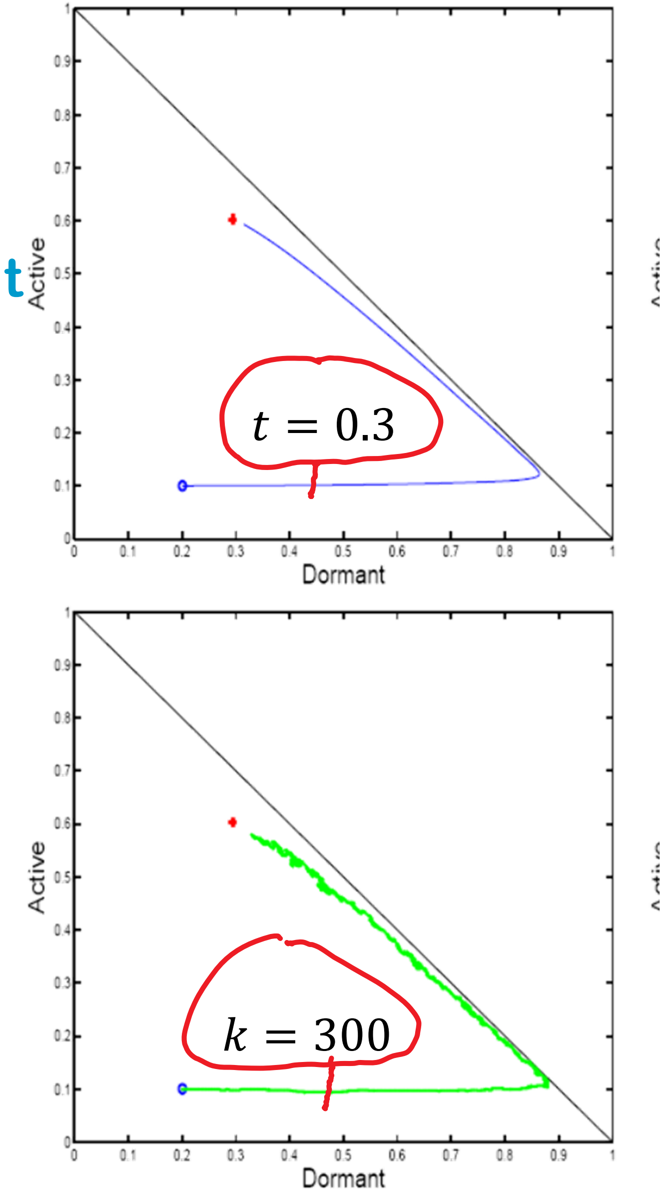
when I(N) = 1/N the condition is true as soon as Second moment of number of objects affected in one timeslot $\leq$ a constant

- Similar result when mean field limit is in discrete time [Le Boudec et al 2007]

# Example: Convergence to Mean Field

**Mean Field Limit**
**N = +∞**

**Stochastic system**
**N = 1000**




- Number of transitions per time step is bounded by 2, therefore there is convergence to mean field
- The Mean field limite is an ODE
- One time step corresponds to $\Delta t = 1/N$

# Formulating the Mean Field Limit

- *Drift* = sum over all transitions of

  proba of transition

  x

  Delta to system state $M^N(t)$

- Re-scale drift by intensity
- Equation for mean field limit is

  dm/dt = limit of rescaled drift

- Can be automated

  http://icawww1.epfl.ch/IS/tsed

| case | prob | effect on $(D, A, S)$ |
|---|---|---|
| 1 | $D\delta_D$ | $\frac{1}{N}(-1, 0, 1)$ |
| 2 | $D\lambda\frac{ND-1}{N-1}$ | $\frac{1}{N}(-2, +2, 0)$ |
| 3 | $A\beta\frac{D}{h+D}$ | $\frac{1}{N}(-1, +1, 0)$ |
| 4 | $A\delta_A$ | $\frac{1}{N}(0, -1, +1)$ |
| 5 | $S(\alpha_0 + rD)$ | $\frac{1}{N}(+1, 0, -1)$ |
| 6 | $S\alpha$ | $\frac{1}{N}(0, +1, -1)$ |

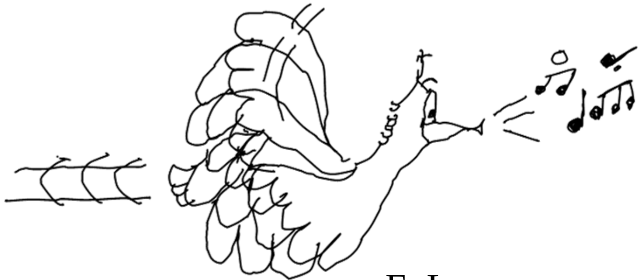
drift =

$$\frac{1}{N}\begin{pmatrix} -D\delta_D - 2D\lambda\frac{ND-1}{N-1} - A\beta\frac{D}{h+D} + S(\alpha_0 + rD) \\ 2D\lambda\frac{ND-1}{N-1} + A\beta\frac{D}{h+D} - A\delta_A + S\alpha \\ D\delta_D + A\delta_A - S(\alpha_0 + rD) - S\alpha \end{pmatrix}$$

ODE

$$\begin{aligned} \frac{\partial D}{\partial t} &= -\delta_D D - 2\lambda D^2 - \beta A\frac{D}{h+D} + (\alpha_0 + rD)S \\ \frac{\partial A}{\partial t} &= 2\lambda D^2 + \beta A\frac{D}{h+D} - \delta_A A + \alpha S \\ \frac{\partial S}{\partial t} &= \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S \end{aligned}$$

# Convergence to Mean Field



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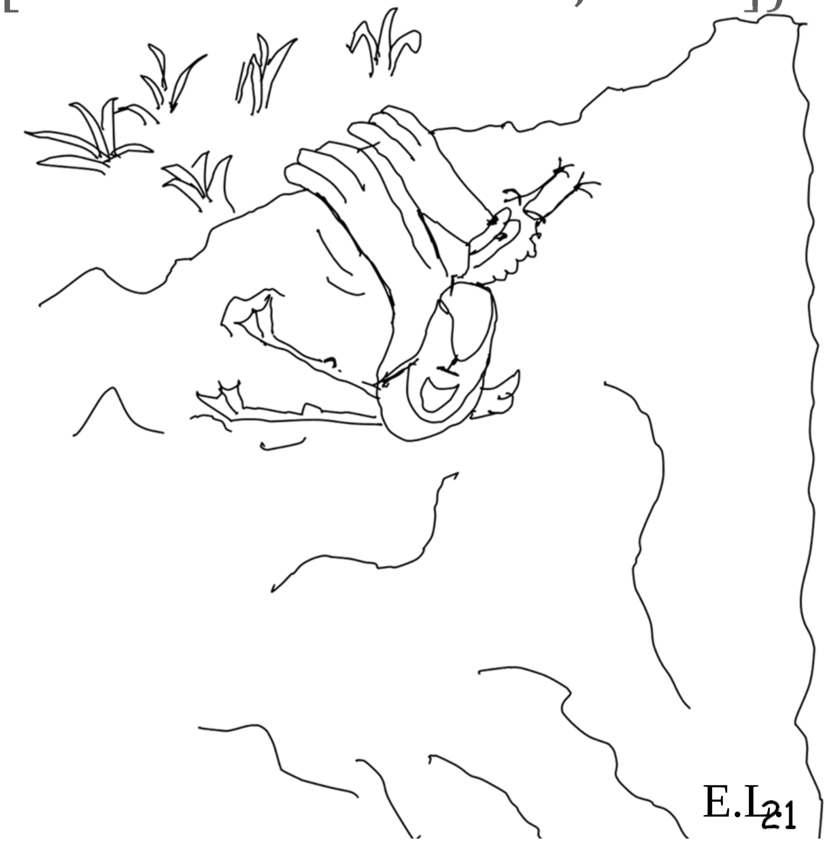
- Thus:

For the finite state space case, most cases are verifiable by inspection of the model

- For the general state space, things may be more complex (fluid limit is not an ODE, e.g. [Chaintreau et al, 2009], [Gomez-Serrano et al, 2012])



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3.

# FINITE HORIZON : FAST SIMULATION AND DECOUPLING ASSUMPTION

# The Decoupling Assumption

- Often used in analysis of complex systems
- Says that $k$ objects are asymptotically mutually independent ($k$ is fixed and $N \to \infty$)
- What is the relation to mean field convergence ?

# The Decoupling Assumption

- Often used in analysis of complex systems
- Says that $k$ objects are asymptotically mutually independent ($k$ is fixed and $N \to \infty$)
- What is the relation to mean field convergence ?

- [Sznitman 1991] [For a mean field interaction model: ]

Decoupling assumption
$\Leftrightarrow$
$M^N(t)$ converges to a deterministic limit

- Further, if decoupling assumption holds, $m(t) \approx$ state proba for any arbitrary object

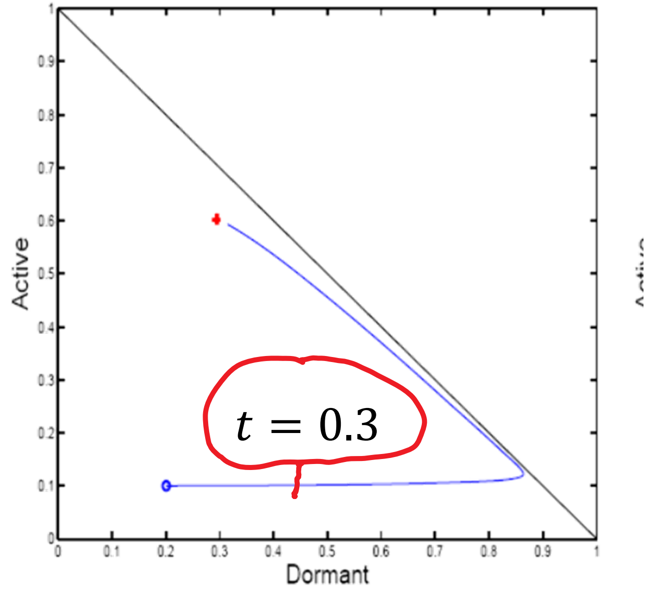
# The Two Interpretations of the Mean Field Limit

- At any time $t$

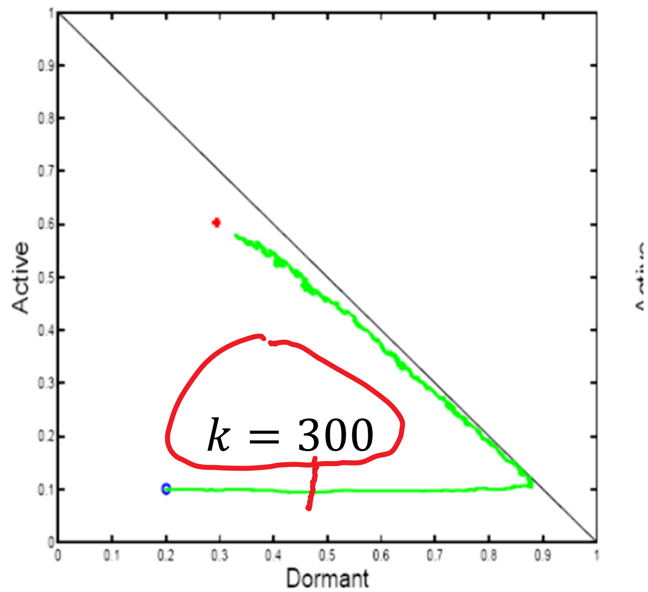
$$P(X_n(t) =' A') \approx A\left(\frac{t}{N}\right)$$

$$P(X_m(t) =' D', X_n(t) =' A') \approx D\left(\frac{t}{N}\right) A\left(\frac{t}{N}\right)$$

where $(D, A, S)$ is solution of ODE




- Thus for $N = 1000$ and simulation step $k = 300$:
  - Prob (node $n$ is dormant) ≈ 0.48
  - Prob (node $n$ is active) ≈ 0.19
  - Prob (node $n$ is susceptible) ≈ 0.33




- *m(t)* approximates both

1. the occupancy measure $M^N(t)$
2. the state probability for one object at time *t*, drawn at random among $N$

# Fast Simulation

- The evolution for one object as if the other objects had a state drawn randomly and independently from the distribution *m(t)*

- Is valid *over finite horizon* whenever mean field convergence occurs

- Can be used to perform «fast simulation», i.e., simulate in detail only one or two objects, replace the rest by the mean field limit (ODE)

## We can fast-simulate one node, and even compute its PDF at any time

$$p_j^N(t|i) = \mathrm{P}(X_n^N(t) = j \,|X_n^N(0) = i)$$

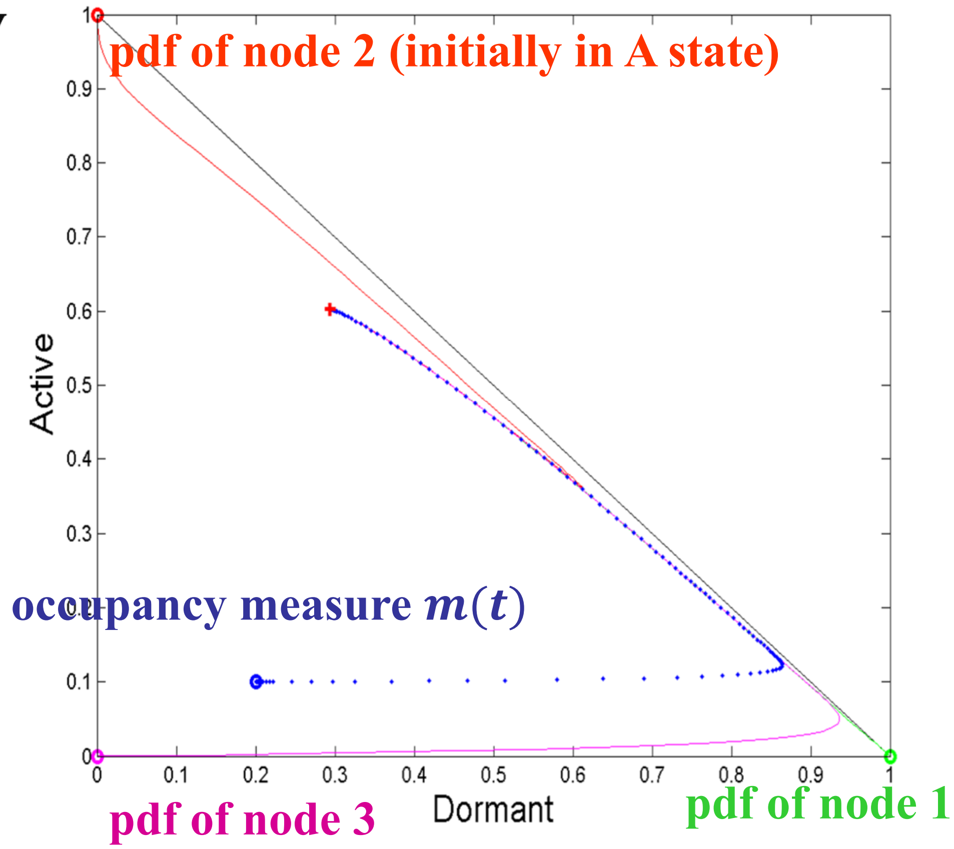
$$p_j^N\left(\frac{t}{N}|i\right) \approx p_j(t|i)$$

where $\vec{p}(t|i)$ is the (transient) probability of a continuous time nonhomogeneous Markov process

$$\frac{d}{dt}\vec{p}(t|i) = \vec{p}(t|i)^T A\big(\vec{m}(t)\big)$$

- Same ODE as mean field limit, with different initial condition

$$\frac{d}{dt}\vec{m}(t) = \vec{m}(t)^T A\big(\vec{m}(t)\big) = F\big(\vec{m}(t)\big)$$



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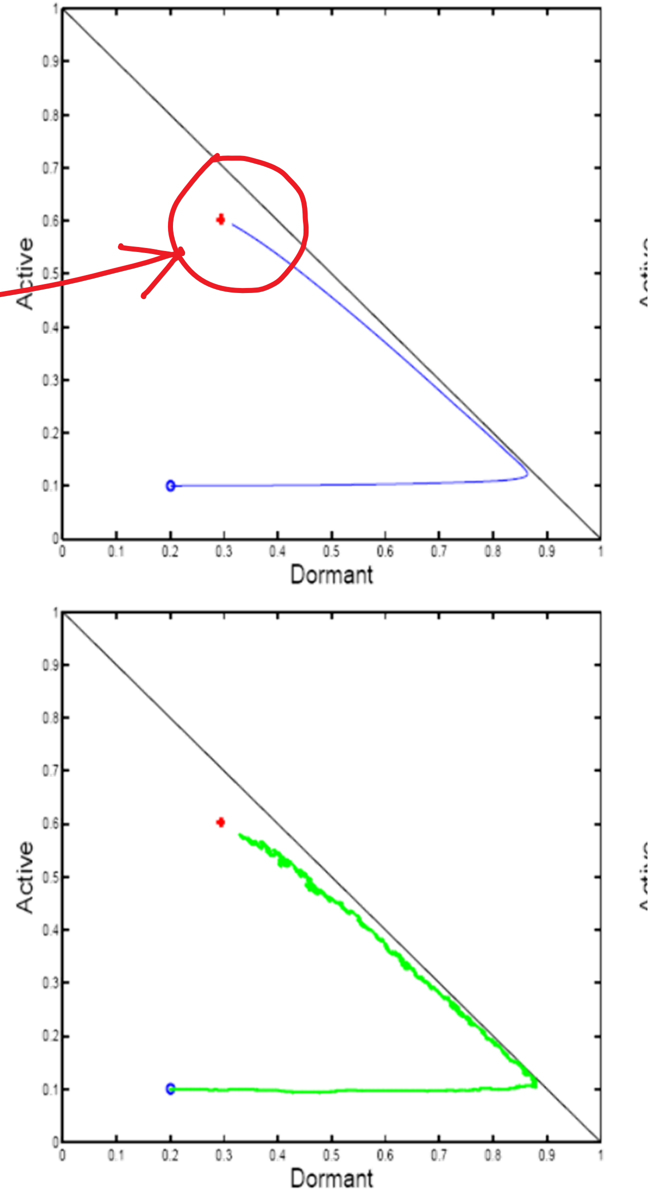
# INFINITE HORIZON: FIXED POINT METHOD AND DECOUPLING ASSUMPTION

# Decoupling Assumption in Stationary Regime

- Stationary regime = for large $t$
- Here:
  - Prob (node $n$ is dormant) ≈ 0.3
  - Prob (node $n$ is active) ≈ 0.6
  - Prob (node $n$ is susceptible) ≈ 0.1
- Decoupling assumption says distribution of prob for state of one object is ≈ $\vec{m}(t)$ with

$$\frac{d\vec{m}(t)}{dt} = F\left(\vec{m}(t)\right)$$

- We are interested in stationary regime, i.e we do $F(\vec{m}) = 0$



# Example: 802.11 Analysis, Bianchi's Formula

802.11 single cell

$m_i$ = proba one node is in backoff stage I

β= attempt rate

γ = collision proba

See [Benaim and Le Boudec , 2008] for this analysis

ODE for mean field limit

$$\frac{dm_0}{d\tau} = -m_0 q_0 + \beta(\vec{m})\left(1-\gamma(\vec{m})\right) + q_K m_K \gamma(\vec{m})$$
$$\frac{dm_i}{d\tau} = -m_i q_i + m_{i-1} q_{i-1} \gamma(\vec{m}) \qquad i = 1, ..., K$$

$$\beta(\vec{m}) = \sum_{i=0}^{K} q_i m_i$$
$$\gamma(\vec{m}) = 1 - e^{-\beta(\vec{m})}$$

**Solve for Fixed Point:**

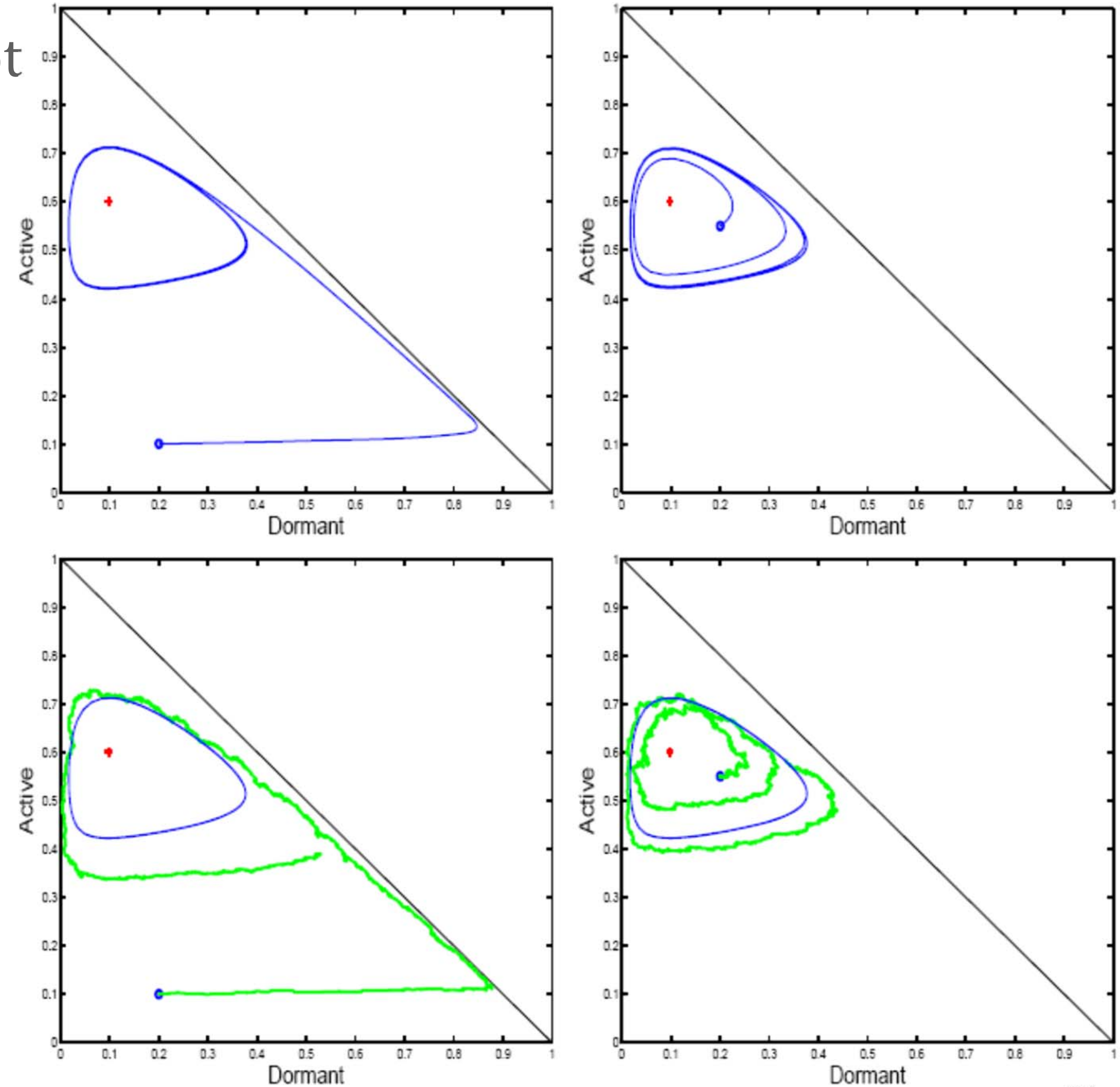
$$m_i = \frac{\gamma^i}{q_i} \frac{1}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

Bianchi's Fixed Point Equation [Bianchi 1998]

$$\gamma = 1 - e^{-\beta}$$
$$\beta = \frac{\sum_{k=0}^{K} \gamma^k}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

# Example Where Fixed Point Method Fails

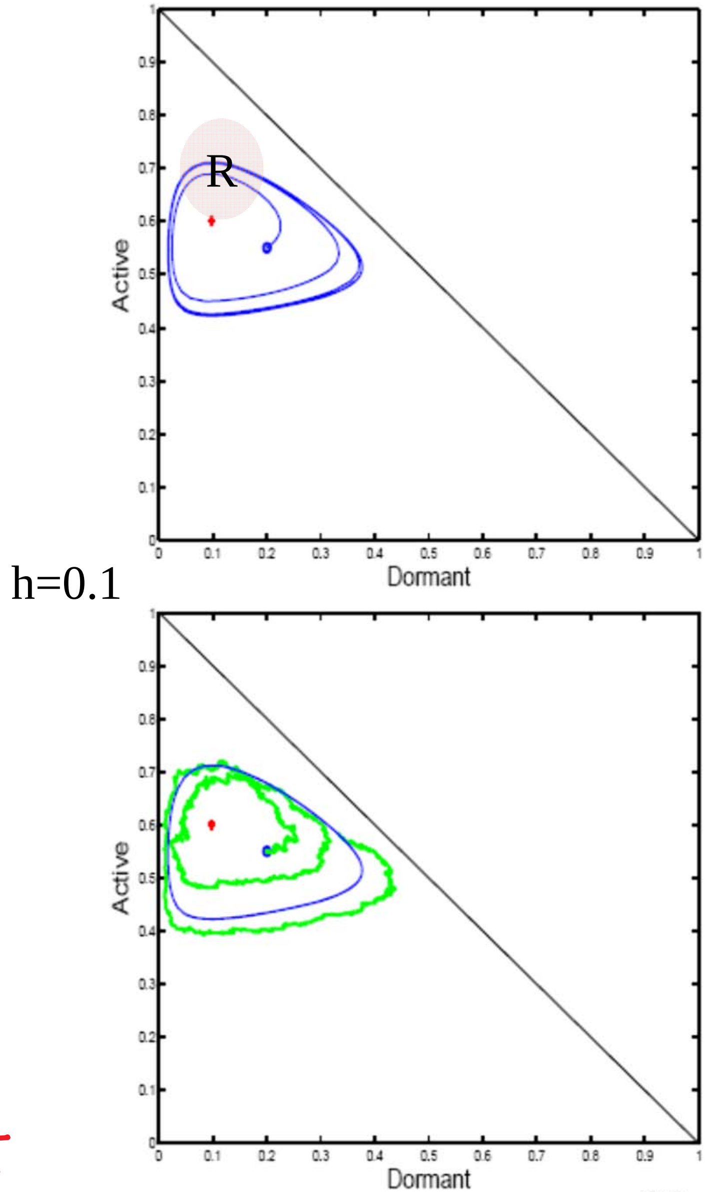
- Same as before except for one parameter value : $h = 0.1$ instead of 0.3
- The ODE does not converge to a unique attractor (limit cycle)
- The equation
$$F(\vec{m}) = 0$$
has a unique solution (red cross) – but it is *not* the stationary regime !



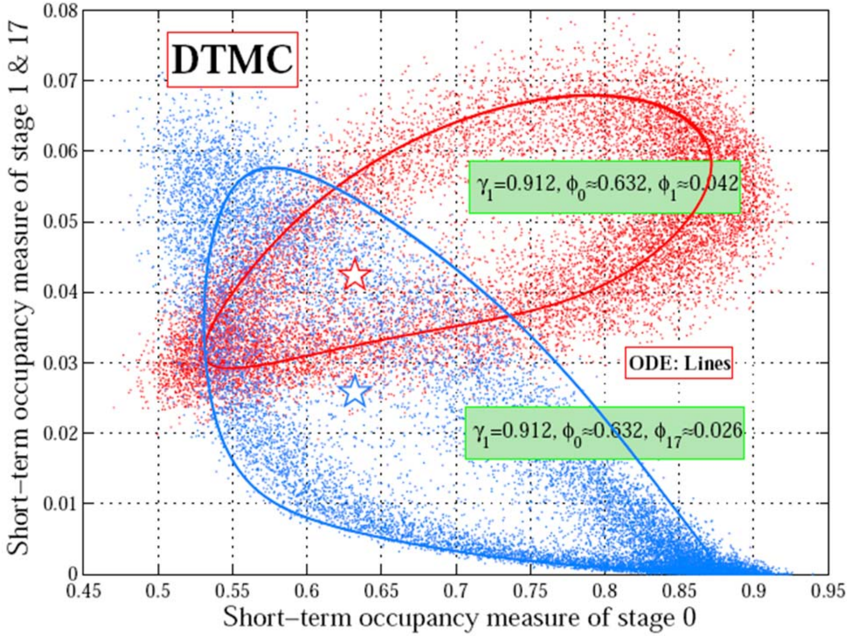
# When the Fixed Point Method Fails, Decoupling Assumption Does not Hold

- In stationary regime, $\vec{m}(t) = (D(t), A(t), S(t))$ follows the limit cycle
- Assume you are in stationary regime (simulation has run for a long time) and you observe that one node, say $n = 1$, is in state 'A'
- It is more likely that $m(t)$ is in region R
- Therefore, it is more likely that some other node, say $n = 2$, is also in state 'A'

- Nodes are not independent – they are *synchronized*



# Example: 802.11 with Heterogeneous Nodes




- [Cho et al, 2010]

  Two classes of nodes with heterogeneous parameters (restransmission probability)

  Fixed point equation has a unique solution, but this is not the stationary proba

  There is a limit cycle

# Where is the Catch ?

- Decoupling assumption says that nodes $m$ and $n$ are asymptotically independent

- There *is* mean field convergence for this example

- But we saw that nodes may not be asymptotically independent

… is there a contradiction ?

Markov chain is ergodic

$$\begin{tikzcd}
\mathbb{P}(X_1^N(t/N) = i \text{ and } X_1^N(t/N) = j) \arrow[rr, "t\to\infty"] \arrow[d, "N\to\infty"'] & & \pi^N_{i,j} \arrow[d, "N\to\infty"] \\
m_i(t)\, m_j(t) & \neq & \frac{1}{T}\int_0^T m_i(t)\, m_j(t)\, dt
\end{tikzcd}$$

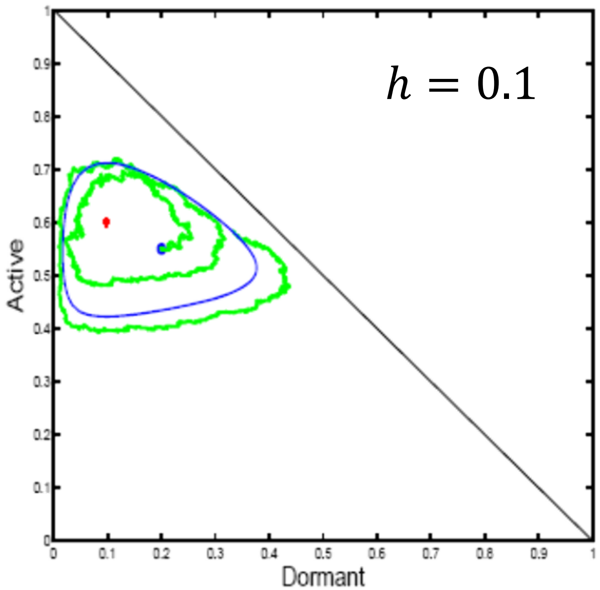
Mean Field Convergence

- The *decoupling assumption may not hold in stationary regime*, even for perfectly regular models

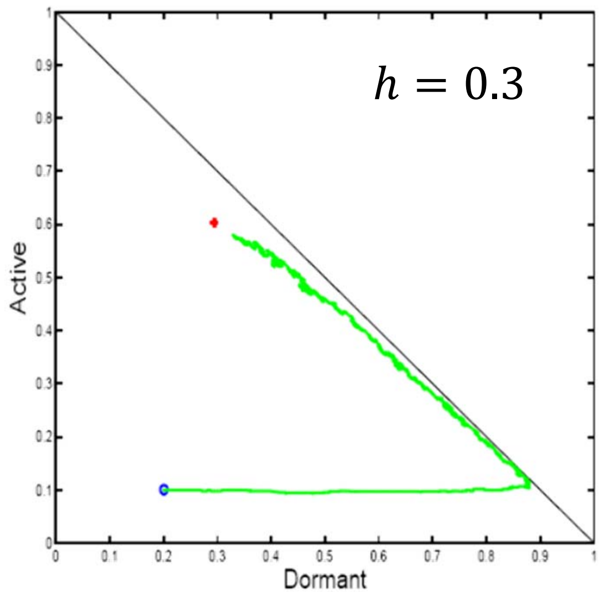
- A correct statement is: conditionally independent given the value of the mean field limit $m(t)$

# Positive Result 1 [e.g. Benaim et al 2008] : Decoupling Assumption Holds in Stationary Regime if mean field limit ODE has a unique fixed point to which all trajectories converge

Decoupling does not hold in stationary regime




Decoupling holds in stationary regime



# Positive Result 2: In the Reversible Case, the Fixed Point Method Always Works

- **Definition** Markov Process $X(t)$ with transition rates $q(i,j)$ is reversible iff

  1. it is ergodic 2. $p(i)\,q(i,j) = p(j)\,q(j,i)$ for some $p$

**Theorem 1.2 ([Le Boudec(2010)])** *Assume some process $Y^N(t)$ converges at any fixed $t$ to some deterministic system $y(t)$ at any finite time. Assume the processes $Y^N$ are reversible under some probabilities $\Pi^N$. Let $\Pi \in \mathcal{P}(E)$ be a limit point of the sequence $\Pi^N$. $\Pi$ is concentrated on the set of stationary points $S$ of the fluid limit $y(t)$*

- Stationary points = fixed points
- If process with finite $N$ is reversible, the stationary behaviour is determined only by fixed points.

# A Correct Method in Order to Make the Decoupling Assumptions

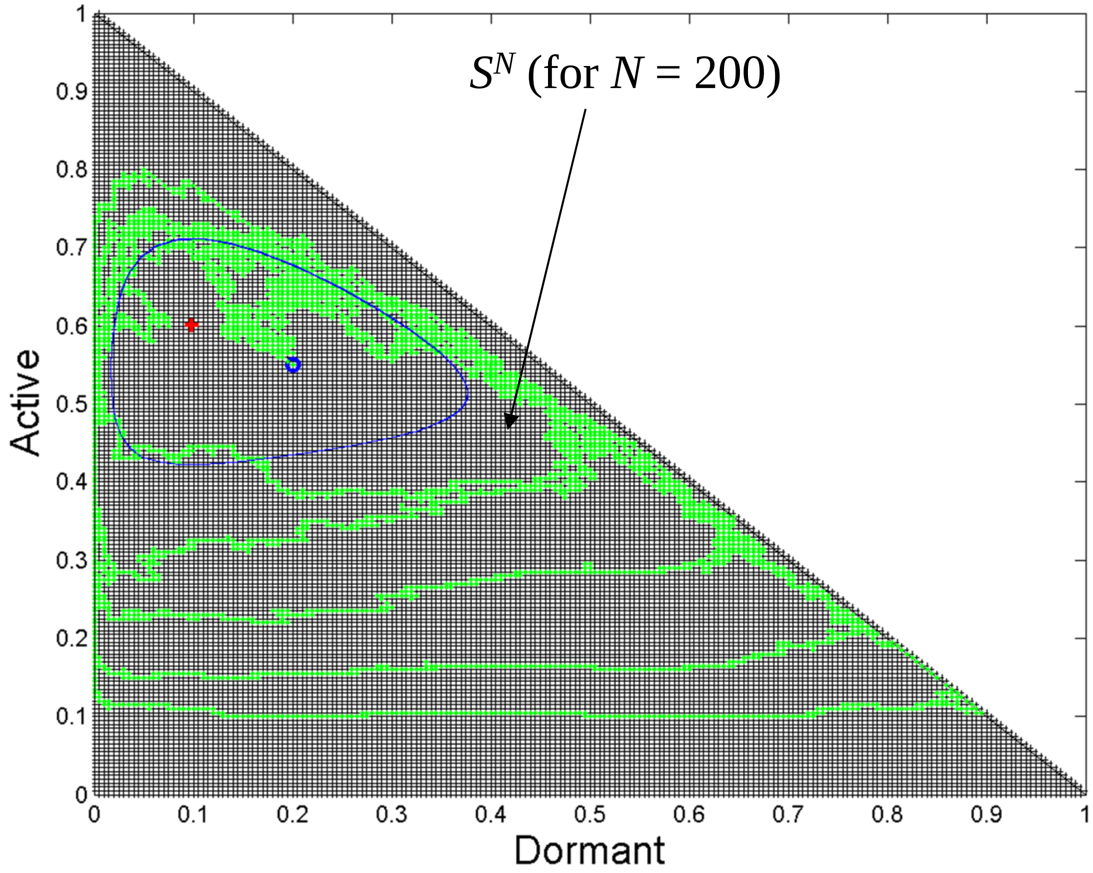
- 1. Write dynamical system equations *in transient regime*

- 2. Study the *stationary regime of* dynamical system
  - **if** converges to unique stationary point $m^*$ **then** make fixed point assumption
  - **else** objects are coupled in stationary regime by mean field limit $m(t)$

- Hard to predict outcome of 2 (except for reversible case)

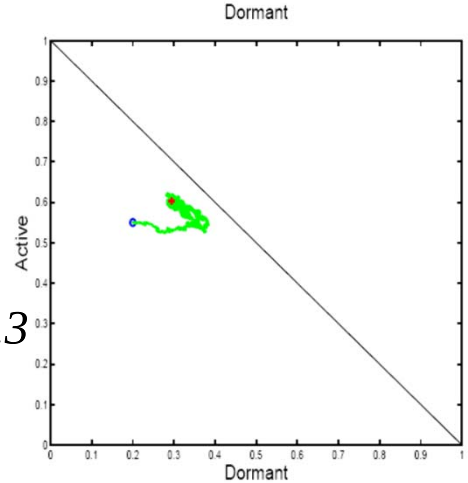
# Stationary Behaviour of Mean Field Limit is not predicted by Structure of Markov Chain

- $M^N(t)$ is a Markov chain on $S^N=\{(a, b, c) \geq 0, a + b + c = 1,\ a, b, c \text{ multiples of } 1/N\}$
- $M^N(t)$ is ergodic and aperiodic, for any value of $h$

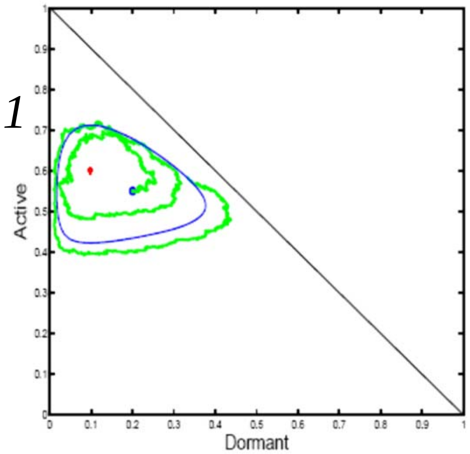



- Depending on $h$, there is or is not a limit cycle for $m(t)$

$h = 0.3$




$h = 0.1$



# Conclusion

- Mean field models are frequent in large scale systems
- Validity of approach is often simple by inspection
- Mean field is both
  - ODE for fluid limit
  - Fast simulation using decoupling assumption
- Decoupling assumption holds at finite horizon; may not hold in stationary regime (except for reversible case)
- Study the stationary regime of the ODE !

  (instead of computing the stationary proba of the Markov chain)

**Thank You …**

# References

[Baccelli et al.(2004)Baccelli, Lelarge, and McDonald] F. Baccelli, M. Lelarge, and D McDonald. Metastable regimes for multiplexed tcp flows. In *Proceedings of the Forty-Second Annual Allerton Conference on Communication, Control, and Computing, Allerton House, Monticello, Illinois, USA, University of Illinois at Urbana-Champaign*, October 2004.

[Benaim and Le Boudec(2008)] M. Benaim and J.Y. Le Boudec. A class of mean field interaction models for computer and communication systems. *Performance Evaluation*, 65(11-12):823–838, 2008.

[Benaïm and Weibull(2003)] M. Benaïm and J. Weibull. Deterministic approximation of stochastic evolution. *Econometrica*, 71:873–904, 2003.

[Benaim et al.(2006)Benaim, Hofbauer, and Sorin] M. Benaim, J. Hofbauer, and S. Sorin. Stochastic approximations and differential inclusions ii: Applications. 2006.

[Bordenave et al.(2007)Bordenave, McDonald, and Proutiere] C. Bordenave, D. McDonald, and A. Proutiere. A particle system in interaction with a rapidly varying environment: Mean field limits and applications. *Arxiv preprint math/0701363*, 2007.

[Bordenave et al.(2008)Bordenave, McDonald, and Proutiere] C. Bordenave, D. McDonald, and A. Proutiere. Performance of random medium access control, an asymptotic approach. In *Proceedings of the 2008 ACM SIGMETRICS international conference on Measurement and modeling of computer systems*, pages 1–12. ACM, 2008.

[Buchegger and Le Boudec(2002)] S. Buchegger and J.-Y. Le Boudec. Performance analysis of the confidant protocol (cooperation of nodes - fairness in dynamic ad-hoc networks). In *Proceedings of MobiHoc'02*, June 2002.

[Chaintreau et al.(2009)Chaintreau, Le Boudec, and Ristanovic] A. Chaintreau, J.Y. Le Boudec, and N. Ristanovic. The age of gossip: spatial mean field regime. In *Proceedings of the eleventh international joint conference on Measurement and modeling of computer systems*, pages 109–120. ACM, 2009.

[Darling and Norris(2008)] RWR Darling and J.R. Norris. Differential equation approximations for Markov chains. *Probability surveys*, 5:37–79, 2008.

[Deffuant et al.(2000)Deffuant, Neau, Amblard, and Weisbuch] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch. Mixing beliefs among interacting agents. *Advances in Complex Systems*, 3:87–98, 2000.

[Ethier and Kurtz(2005)] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes, Characterization and Convergence*. Wiley, 2005.

[Gast et al.(2010)Gast, Gaujal, and Le Boudec] Nicolas Gast, Bruno Gaujal, and Jean-Yves Le Boudec. Mean field for Markov Decision Processes: from Discrete to Continuous Optimization. Technical Report arXiv:1004.2342v2, 2010.

[Gomez-Serrano et al, 2012] Gomez-Serrano J., Graham C. and Le Boudec J.-Y.
The Bounded Confidence Model Of Opinion Dynamics
*Mathematical Models and Methods in Applied Sciences*, Vol. 22, Nr. 2, pp. 1150007-1--1150007-46, 2012.

[Graham and Méléard(1994)] Carl Graham and Sylvie Méléard. Chaos hypothesis for a system interacting through shared resources. *Probab. Theory Related Fields*, 100(2):157–173, 1994. ISSN 0178-8051. doi: 10.1007/BF01199263. URL `http://dx.doi.org/10.1007/BF01199263`.

[Graham and Méléard(1997)] Carl Graham and Sylvie Méléard. Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.*, 25(1):115–132, 1997. ISSN 0091-1798. doi: 10.1214/aop/1024404281. URL `http://dx.doi.org/10.1214/aop/1024404281`.

[Ioannidis and Marbach(2009)] S. Ioannidis and P. Marbach. Absence of Evidence as Evidence of Absence: A Simple Mechanism for Scalable P2P Search. In *INFOCOM 2009, IEEE*, pages 576–584. IEEE, 2009.

[Kelly(1991)] F.P. Kelly. Loss networks. *The annals of applied probability*, 1 (3):319–378, 1991.

[Kurtz(1970)] T.G. Kurtz. Solutions of ordinary differential equations as limits of pure jump Markov processes. *Journal of Applied Probability*, 7(1): 49–58, 1970.

[Le Boudec(2010)] Jean-Yves Le Boudec. The Stationary Behaviour of Fluid Limits of Reversible Processes is Concentrated on Stationary Points. Technical Report arxiv:1009.5021.v2, 2010.

[Le Boudec et al.(2007)Le Boudec, McDonald, and Mundinger] Jean-Yves Le Boudec, David McDonald, and Jochen Mundinger. A Generic Mean Field Convergence Result for Systems of Interacting Objects. In *QEST'07*, 2007.

[McDonald(2007)] David McDonald. Lecture Notes on Mean Field Convergence, March 2007.

[Sandholm(2006)] W.H. Sandholm. Population games and evolutionary dynamics. *Unpublished manuscript, University of Wisconsin*, 2006.

[Sznitman(1991)] A.S. Sznitman. Topics in propagation of chaos. In P.L. Hennequin, editor, *Springer Verlag Lecture Notes in Mathematics 1464, Ecole d'Eté de Probabilités de Saint-Flour XI (1989)*, pages 165Ű–251, 1991.

[Tembine et al.(2009)Tembine, Le Boudec, El-Azouzi, and Altman] Hamidou Tembine, Jean-Yves Le Boudec, Rachid El-Azouzi, and Eitan Altman. Mean Field Asymptotic of Markov Decision Evolutionary Games and Teams. In *Gamenets 2009*, 2009. Invited Paper.

[Tinnakornsrisuphap and Makowski(2003)] Peerapol Tinnakornsrisuphap and Armand M. Makowski. Limit behavior of ecn/red gateways under a large number of tcp flows. In *Proceedings IEEE INFOCOM 2003, The 22nd Annual Joint Conference of the IEEE Computer and Communications Societies, San Francisco, CA, USA*, March 30 - April 3 2003.