Distribution of Age of Information: A Palm Calculus Approach

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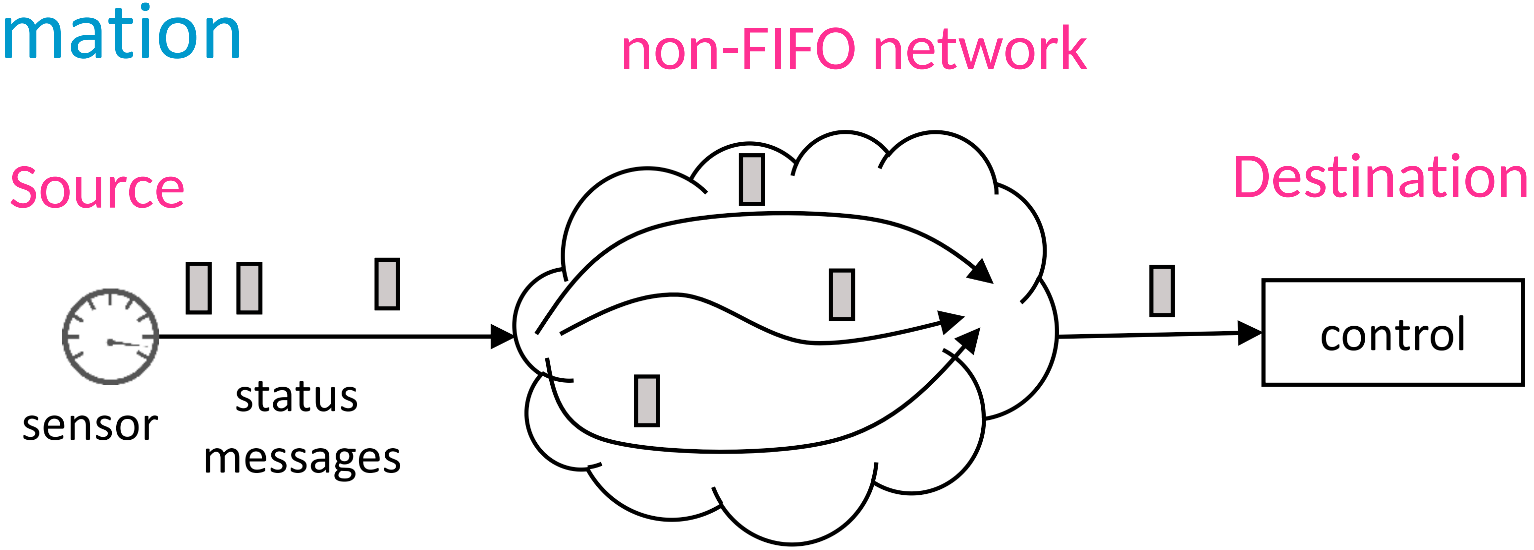
Abstract: In latency-sensitive networked systems, a key metric is the age of information, defined as the time elapsed since the generation of the last received informative status message, measured at an arbitrary point in time. We consider non-preemptive and non-FIFO systems, where messages can overtake each other and the reception of informative messages may obsolete some earlier messages that are underway. Specifically, we compute the distribution of the age of information for a system with Poisson generation of messages, exponential transit times and non FIFO channel. We use Palm calculus and time reversal applied to Markov processes.

[Rizk and Le Boudec 2023] Rizk, A. and Le Boudec, J.Y., 2023. A Palm calculus approach to the distribution of the age of information. *IEEE Transactions on Information Theory*, *69*(12), pp.8097-8110.

Age of Information

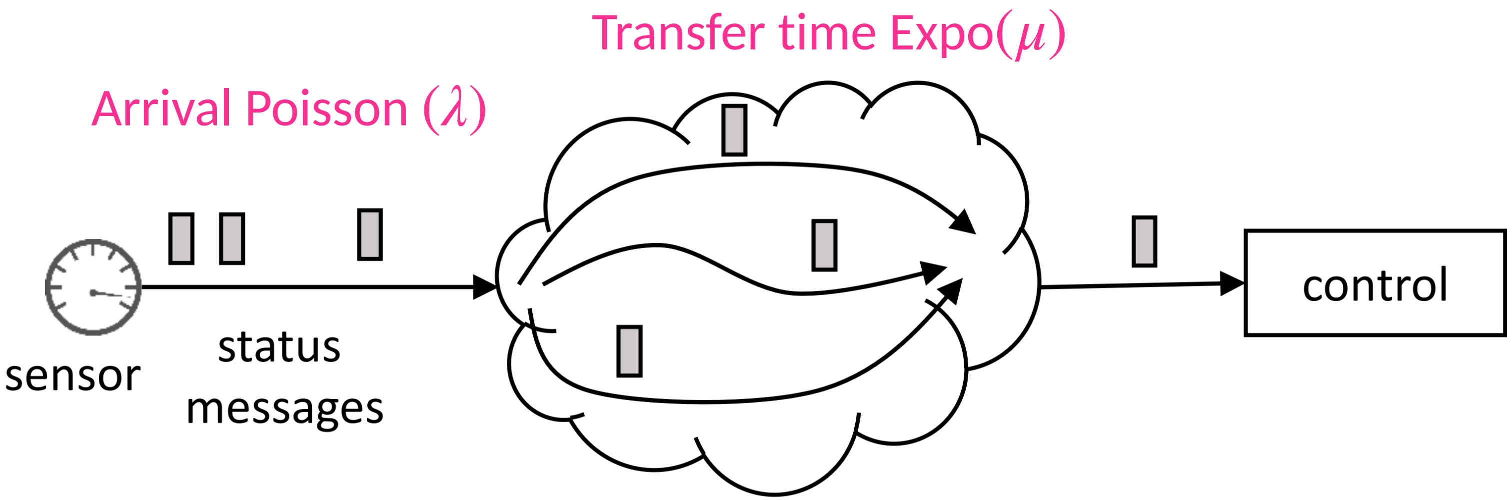


- Time-stamped messages are sent according to a Poisson process of rate λ
- Messages are transferred over a non-FIFO network;
  - message transfer times are iid Exponential (μ)
  - overtaking is possible
- Destination is interested only about latest message
  - messages received *after* a more recent one are discarded
  - source and destination are time-synchronized

age of a message received at destination = now — timestamp in message

Age of information = age of the most recent message received at destination

M/M/$I_{\max}$/$I^*_{\max}$ Queue



- An in-flight message is *informative* if its emission time is after the emission time of the most recently received message.
  In-flight messages that are not informative will be discarded at destination.
- **Assumption**: number of informative in-flight messages $\leq I_{\max}$

  When the number of informative in-flight reaches $I_{\max}$, source refrains from sending

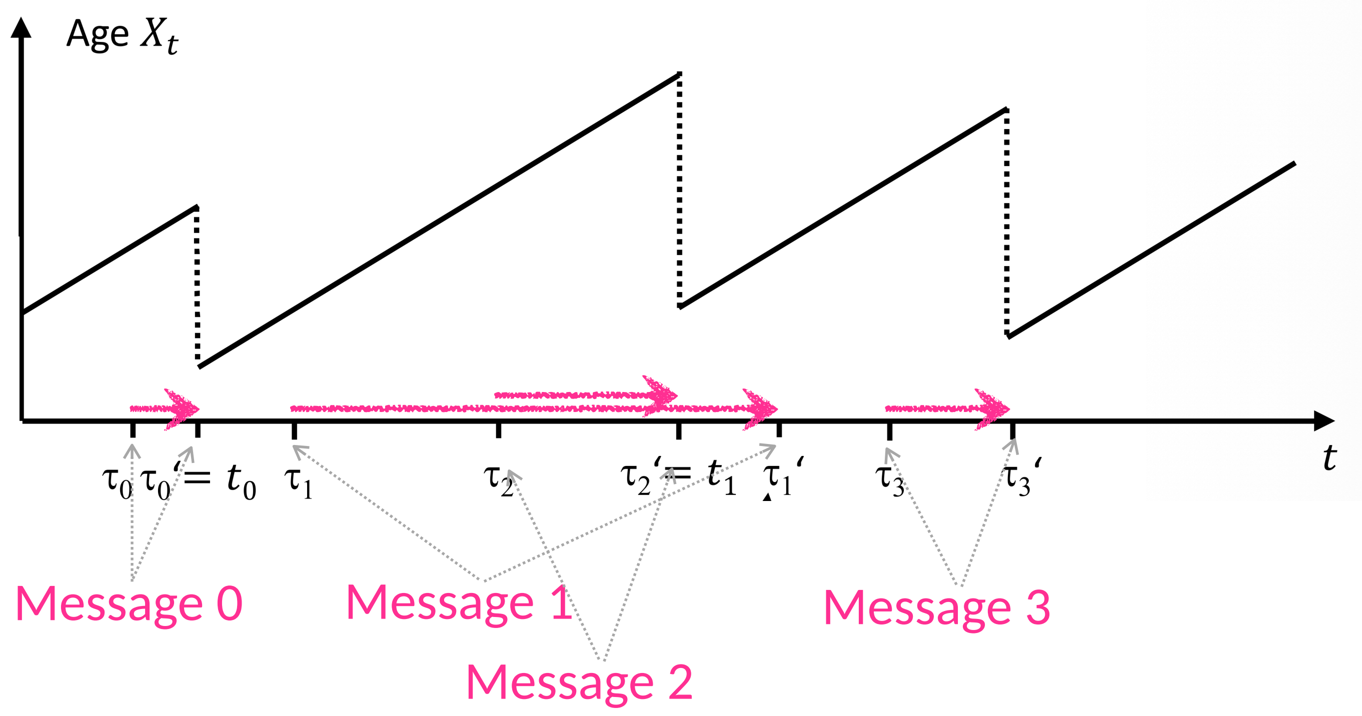
Age of Information Process X_t

$(\tau_j)_{j\in\mathbb{N}}$ = emission times

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots$$

$(\tau'_j)_{j\in\mathbb{N}}$ = reception times

$$\tau_j < \tau'_j$$



$$X_t = t - \max_{j:\tau'_j \le t} \tau_j$$

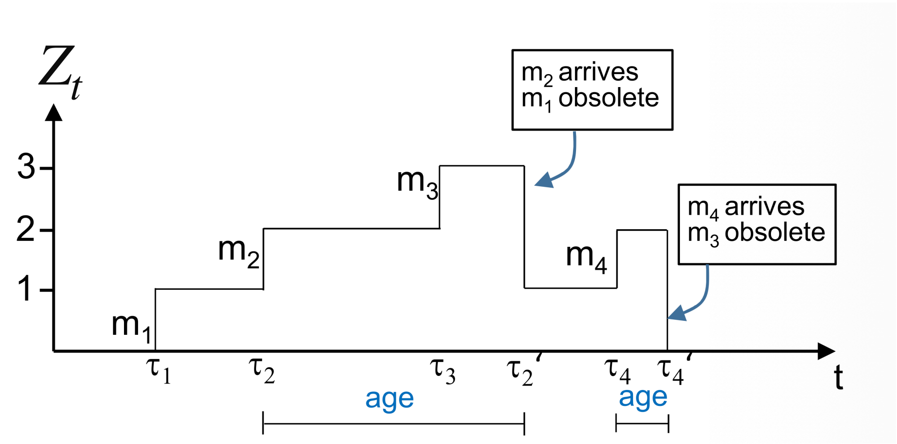
Message 1 is non informative and is dropped at destination

We are interested in the distribution of X_t in stationary regime.

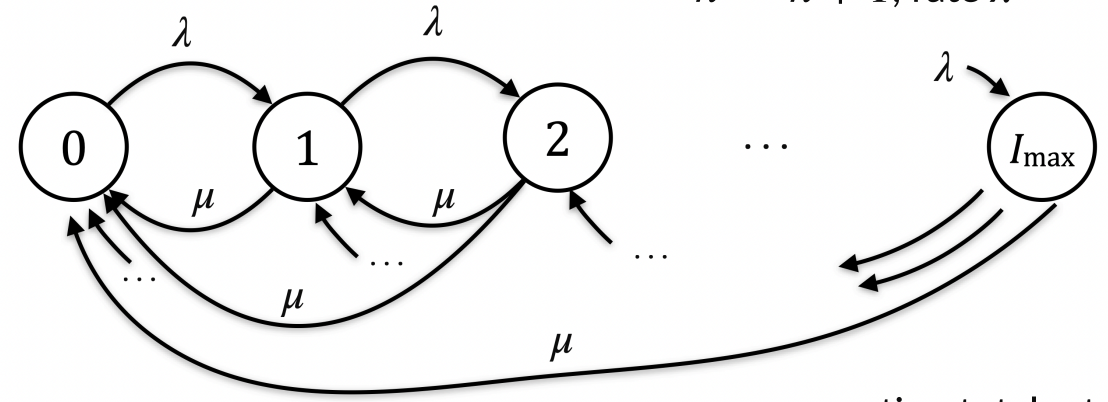
The process of informative in-flight messages

Let $Z_t \in \{0,1,\ldots,I_{\max}\}$ be the number of informative in-flight messages (the "queue").

Z_t is a continuous time Markov chain on the finite set $\{0,1,\ldots,I_{\max}\}$.

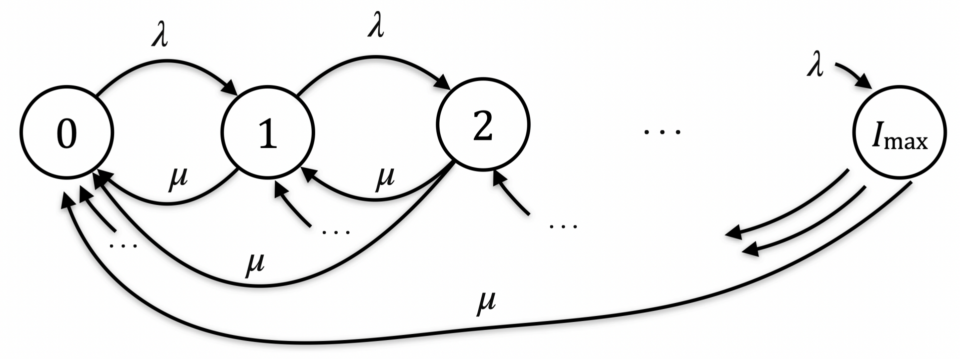


message generation:
$n \to n+1$, rate λ



message reception total rate $n\mu$
a message reception removes this + all non informative messages
$n \to n-i$, rate μ

Stationary Behaviour of Z_t



Z_t is ergodic (finite state space, fully connected).

Balance equation for stationary probability p_n :

$$p_n(\lambda + n\mu) = p_{n-1}\lambda + \sum_{i=n+1}^{I_{\max}} p_i \mu$$

Can be solved explicitly:

$$p_n = \frac{(n+1)\lambda^n \mu}{\prod_{j=1}^{n+1}(\lambda + j\mu)}, \quad 0 \leq n < I_{\max}, \qquad p_{I_{\max}} = \frac{\lambda^{I_{\max}}}{\prod_{j=1}^{I_{\max}}(\lambda + j\mu)}$$

Goal: obtain PDF of Age of Information X_t

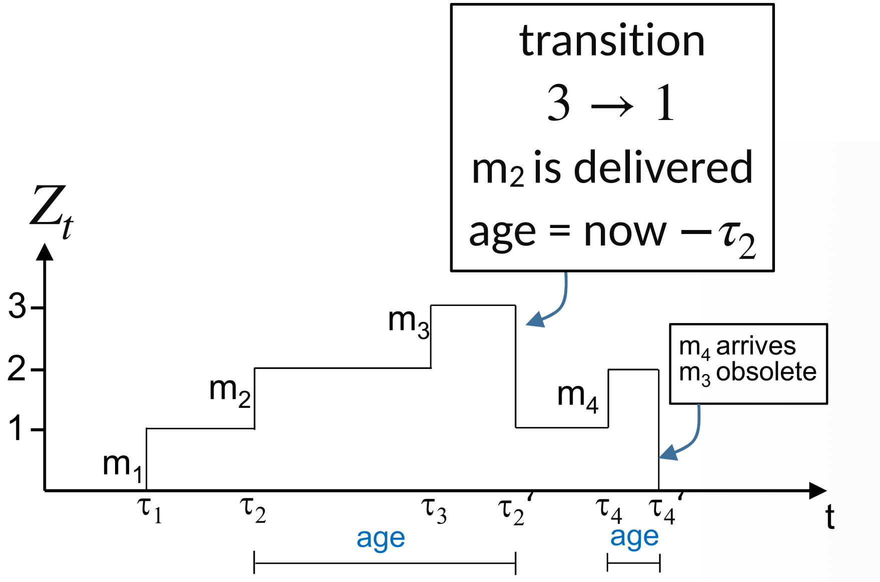
(Z_t, X_t) is a jump and drift process on $\{0,1,\ldots,I_{\max}\} \times \mathbb{R}^+$

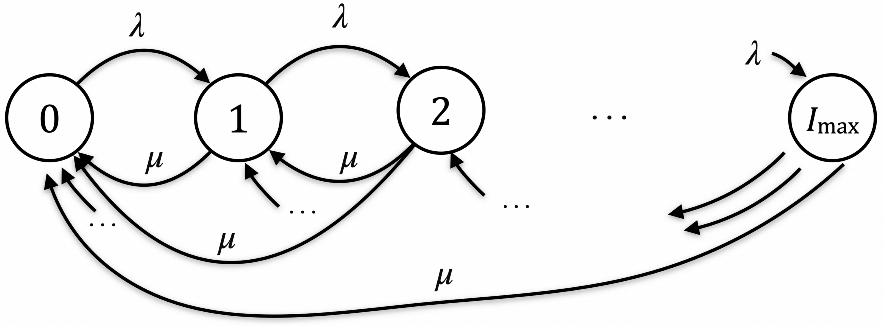
Can be studied as the projection of a Markov drift and jump process [Yates et al, 2021]. Provides stationary mean value of X_t

Here we follow a different approach to obtain full PDF

- Age X_t is updated at some jump instants of Z_t
- X_t is a deterministic function of Z_t's trajectory
- We assume queue Z_t is in stationary regime

⇒ Palm calculus





[Yates et al 2021] Yates, R.D., Sun, Y., Brown, D.R., Kaul, S.K., Modiano, E. and Ulukus, S., 2021. Age of information: An introduction and survey. *IEEE Journal on Selected Areas in Communications*, *39*(5), pp.1183-1210.

Palm Calculus : Framework

A stationary process with state Z_t.

Some random quantity X_t measured at time t. Assume that

(Z_t, X_t) is jointly stationary

E.g. Z_t is in a stationary regime and X_t depends on the past, present and future state of Z_t in a way that is invariant by shift of time origin.

Z_t and X_t are right-continuous.

- Jointly stationary: X_t = age of information
- Not jointly stationary: date of emission of most current message

In our case, Z_t is Markov, (Z_t, X_t) need not be.

[Baccelli and Brémaud, 2012] Baccelli, F. and Brémaud, P., 2012. *Palm probabilities and stationary queues* (Vol. 41). Springer Science & Business Media.

[Le Boudec 2010] Le Boudec, J.Y., 2010. *Performance evaluation of computer and communication systems*. Lausanne: EPFL Press. Available online: https://leboudec.github.io/perfeval/

The Arbitrary Point in Time

When (Z_t, X_t) is jointly stationary, $\mathbb{E}\left(X_t\right)$ is the same at all t

It represents the average seen at an arbitrary point in time

Is also the average seen by an external observer who observes the system at a random time, sampled from a Poisson process of any rate, independent of X_t

(PASTA: Poisson Arrivals See Time Averages)

Stationary Point Process

Z_t is a Markov chain on some discrete state space $\mathcal{S}$

Let $\mathcal{F}_0 \subset \mathcal{S} \times \mathcal{S}$ be a subset of selected transitions of Z_t

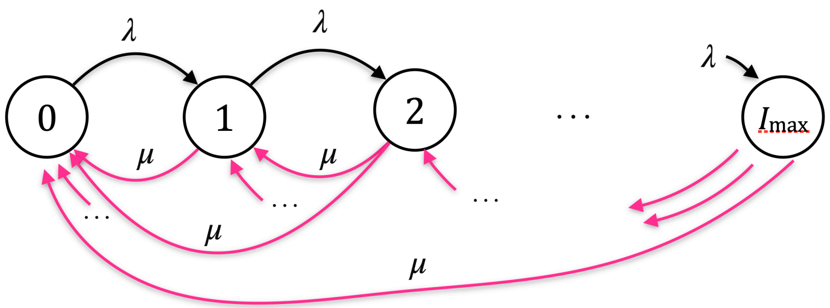
Let T_n be the sequence of instants at which transitions in $\mathcal{F}_0$ occur.

T_n is a stationary point process associated with Z_t

By convention, in Palm calculus:

$$\ldots < T_{-2} < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \ldots$$

and time $t = 0$ is the *arbitrary* point in time.



Example: $T_n = \tau'_n$ = message delivery instants in Z_t

$\mathcal{S} = \{0, 1, \ldots, I_{\max}\}$

$\mathcal{F}_0 = \{(n, n-i) \in \mathcal{S}^2 \mid 1 \leq i \leq n\}$

Intensity of point process λ = expected number of instants T_n per time unit

$$= \frac{1}{b-a}\mathbb{E}\left(\sum_{n=-\infty}^{+\infty} 1_{a<T_n\leq b}\right)$$

Palm Expectation = Event average

Assume: Z_t, X_t, jointly stationary, T_n is a stationary point process associated with Z_t

Definition : the Palm Expectation is

$$\mathbb{E}^t(X_t) = \mathbb{E}(X_t \,|\, t = T_n \text{ for some } n)$$

By stationarity:

$$\mathbb{E}^t(X_t) = \mathbb{E}^0(X_0)$$

Example:

$T_n =$ instant of delivery of informative message

$X_t =$ age of information at time t

$\mathbb{E}^t(X_t) = \mathbb{E}^0(X_0) =$ average age of information juste after arrival of informative message

Formal Definition

In discrete time, the Palm expectation is an elementary conditional probability

$$\mathbb{E}^t(X_t) = \mathbb{E}(X_t \,|\, \exists n, T_n = t) = \mathbb{E}(X_t \,|\, N_t = 1) = \frac{\mathbb{E}(X_t N_t)}{\mathbb{E}(N_t)} = \frac{\mathbb{E}(X_t N_t)}{\mathbb{P}(N_t = 1)}$$

where $N_t = 1_{\exists n, T_n = t}$

In continuous time, the definition uses Radon-Nikodym derivative

Palm probability is defined similarly

$$\mathbb{P}^0(X_0 \in W) = \mathbb{P}(X_0 \in W \,|\, \exists n, T_n = 0)$$

Note that $\mathbb{P}^0\left(T_0 = 0\right) = 1$ i.e. under the Palm probability, $T_0 = 0$ certainly.

Ergodic Interpretation

For a stationary and ergodic case we can estimate expectations by long-run averages:

Expectation (time average):

$$\mathbb{E}(X_0) = \mathbb{E}(X_t) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T X_s ds$$

Palm expectation (event average):

$$\mathbb{E}^0(X_0) = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} X_{T_n}$$

Palm Probability for Markov Chains

Z_t is a continuous-time, ergodic Markov chain on some discrete state space $\mathcal{S}$ with transition matrix A and stationary probability $(p_i)_{i\in\mathcal{S}}$

Let $\mathcal{F}_0 \subset \mathcal{S} \times \mathcal{S}$ be a subset of selected transitions of Z_t

Let T_n be the sequence of instants at which selected transitions occur.

Theorem: 1. The intensity of the point process T_n is $\hat{\lambda} = \sum_{(i,j)\in\mathcal{F}_0} p_i A_{i,j}$

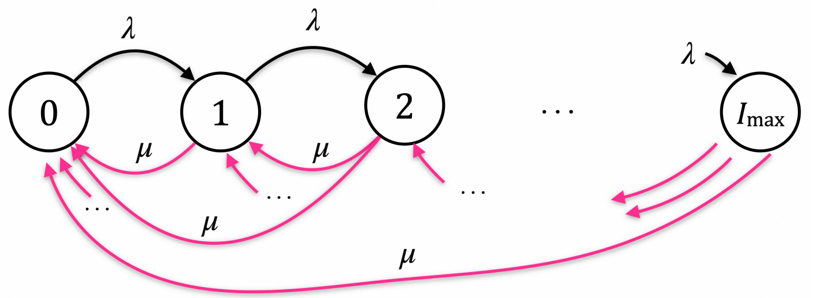
2. The probability that an arbitrary selected transition is (i, j) is

$$\mathbb{P}^0(Z_{0^-} = i, Z_0 = j) = \frac{1}{\hat{\lambda}} 1_{(i,j)\in\mathcal{F}_0} p_i A_{i,j}$$

3. The probability to be in state j just after an arbitrary selected transition is

$$\mathbb{P}^0(Z_0 = j) = \frac{1}{\hat{\lambda}} \sum_{i:(i,j)\in\mathcal{F}_0} p_i A_{i,j}$$

[Le Boudec 2010] Le Boudec, J.Y., 2010. Performance evaluation of computer and communication systems. Lausanne: EPFL Press. Available online: https://leboudec.github.io/perfeval/



Example: $T_n = \tau'_n$ = message delivery instants in Z_t

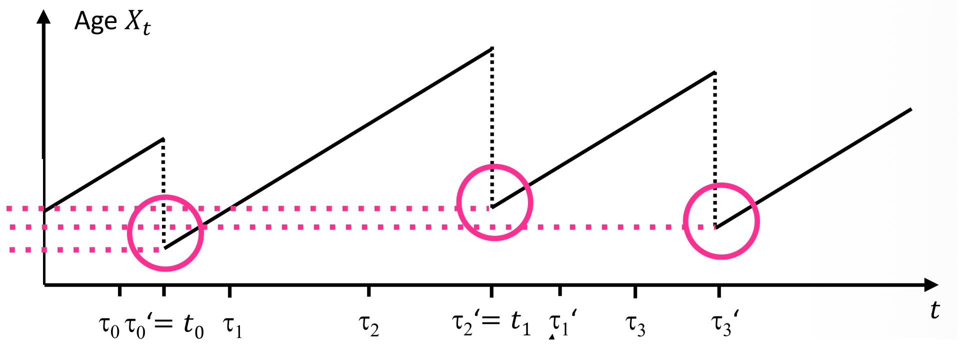
$$\hat{\lambda} = \mu \sum_{i=0}^{I_{max}} i p_i = \mu \bar{Z}$$

$$p^0_{n,n-i} = \mathbb{P}^0(Z_{0^-} = n, Z_0 = n - i) = \frac{1}{\hat{\lambda}} p_n \mu = \frac{p_n}{\bar{Z}}$$

for $1 \leq i \leq n$

Age of Information at Arrival of Informative Messages

Pick an arbitrary arrival instant of informative message. What is the distribution of the age X_t just after this arrival ?

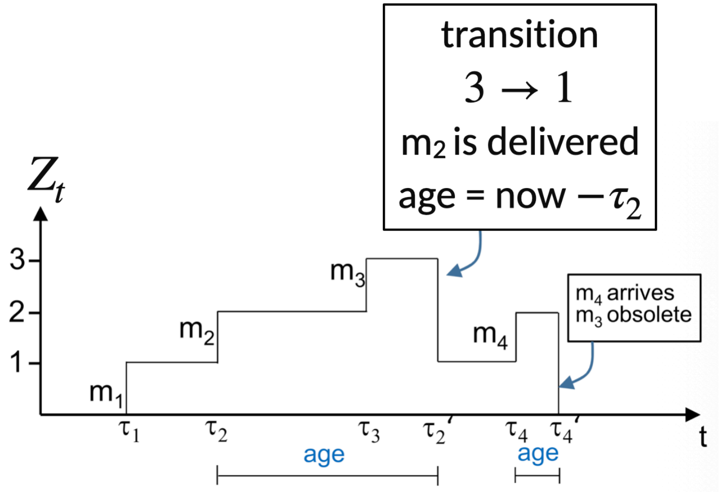


Its Laplace-Stieltjes transform is $\tilde{f}_A(\theta) = \mathbb{E}^0\left(e^{-\theta X_0}\right)$

Condition on the transition at this event being $(n, n-i)$:

$$\tilde{f}_A(\theta) = \mathbb{E}^0\left(e^{-\theta X_0}\right) = \sum_{n,i} p^0_{n,n-i} \mathbb{E}^0\left(e^{-\theta X_0} \middle| X_{0^-=n}, X_0 = i\right)$$

$$p^0_{n,n-i} \rightarrow \frac{p_n}{\bar{Z}} \qquad \mathbb{E}^0\left(e^{-\theta X_0} \middle| X_{0^-=n}, X_0 = i\right) = \tilde{f}_{n,n-i}(\theta)$$



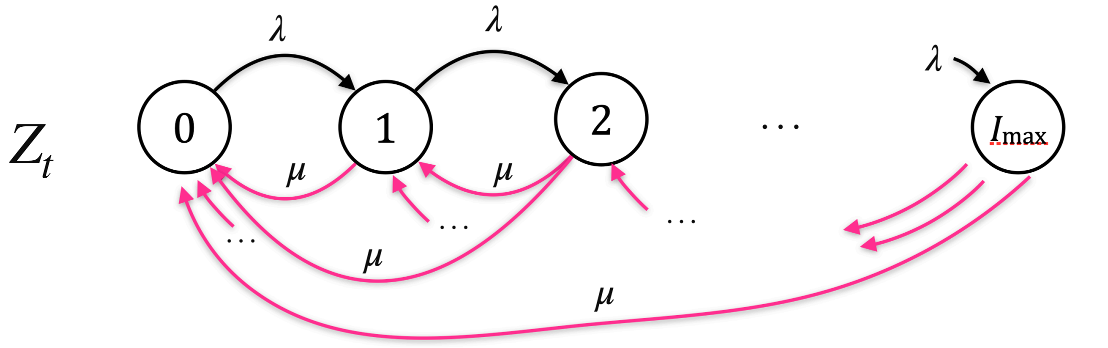
Need to compute $\tilde{f}_{n,n-i}(\theta)$ = LST of time elapsed since i^{th} most recent arrival, given Z_t just did transition $(n, n-i)$

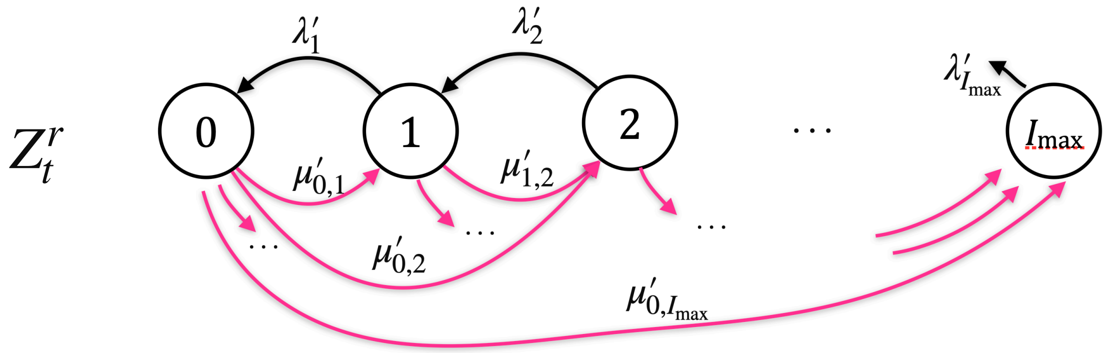
Time Reversal

To compute $\tilde{f}_{n,n-i}(\theta)$ we use the time-reversed process $Z_t^r = Z_{-t^-}$

Z_t^r is a queue with batch arrivals and single departures and state-dependent arrival and service rates:

[Kelly 1979]: Z_t^r is time-homogeneous Markov with transition matrix $A'_{n-i,n} = A_{n,n-i}\dfrac{p_n}{p_{n-i}}$





$$\lambda'_i = \begin{cases} \frac{i}{i+1}\left(\lambda + (i+1)\mu\right) & \text{for } 1 \le i < I_{\max} \\ I_{\max}\mu & \text{for } i = I_{\max} \end{cases}$$

$$\mu'_{ij} = \begin{cases} \dfrac{(j+1)\mu\lambda^{j-i}}{(i+1)\prod\limits_{k=i+2}^{j+1}(\lambda+k\mu)} & \text{for } j \neq I_{\max} \\ \dfrac{\lambda^{I_{\max}-i}}{(i+1)\prod\limits_{k=i+2}^{I_{\max}}(\lambda+k\mu)} & \text{for } j = I_{\max} \end{cases}$$

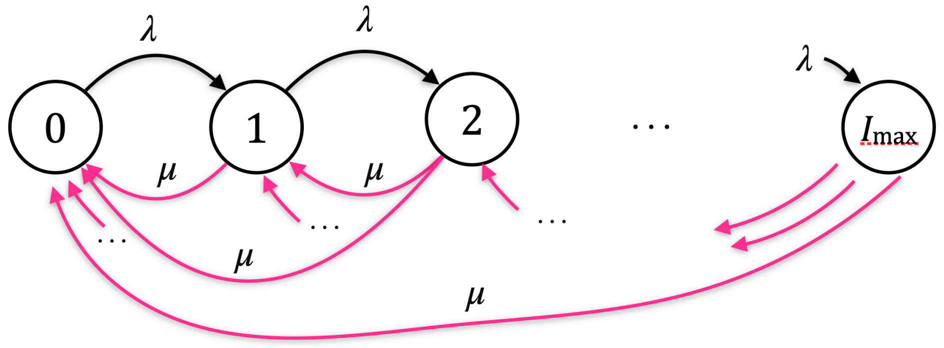
[Kelly 1979] Kelly, F.P., 1979. *Reversibility and stochastic networks* (p. 21). J. Wiley.

Time Reversal

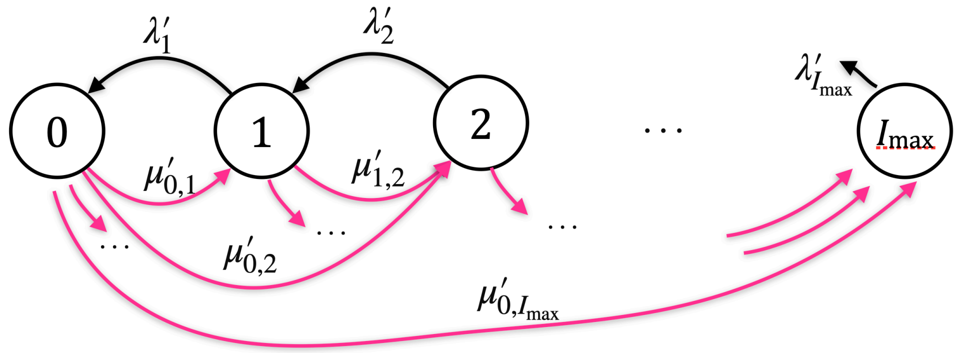
(Coupling:) time elapsed since i^{th} most recent arrival, given Z_t just does transition $(n, n-i)$ = time until next i^{th} departure given Z_t^r just does a transition $(n-i, n)$

(Markov:) $\tilde{f}_{n,n-i}(\theta)$ = LST of time until next i^{th} departure in Z_t^r given $Z_t^r = n$

Z_t



Z_t^r



$$\lambda_i' = \begin{cases} \frac{i}{i+1}(\lambda + (i+1)\mu) & \text{for } 1 \le i < I_{\max} \\ I_{\max}\mu & \text{for } i = I_{\max} \end{cases}$$

$$\mu_{ij}' = \begin{cases} \frac{(j+1)\mu\lambda^{j-i}}{(i+1)\prod_{k=i+2}^{j+1}(\lambda+k\mu)} & \text{for } j \ne I_{\max} \\ \frac{\lambda^{I_{\max}-i}}{(i+1)\prod_{k=i+2}^{I_{\max}}(\lambda+k\mu)} & \text{for } j = I_{\max} \end{cases}$$

Time Until Next Departure

Lemma 4: Let $\tilde{Z}_t$ be a time-homogeneous Markov chain on finite state $\mathcal{S}$ and transition matrix A, and let $\tilde{\mathcal{F}} \subset \mathcal{S} \times \mathcal{S}$. The LST $\tilde{g}_{n,i}$ of time until next i^{th} transition in $\tilde{\mathcal{F}}$ given current state is n satisfies

$$\tilde{g}_{n,i}(\theta) = \frac{1}{\sum_{n' \in \mathcal{S}} A_{n,n'} + \theta} \left(\sum_{n':(n,n') \notin \tilde{\mathcal{F}}} A_{n,n'} \tilde{g}_{n',i}(\theta) + \sum_{n':(n,n') \in \tilde{\mathcal{F}}} A_{n,n'} \tilde{g}_{n',i-1}(\theta) \right)$$

where $\tilde{g}_{n',0}(\theta) = 1$ for all n'

⇒We obtain $\tilde{f}_{n,n-i}(\theta)$ in matrix-geometric form from: $\mathbf{f}_{\cdot,\mathbf{k}} = \boldsymbol{\Psi}^{\mathbf{k}-\mathbf{1}} \boldsymbol{\Phi}^{-\mathbf{1}} \bar{\boldsymbol{\lambda}}'$

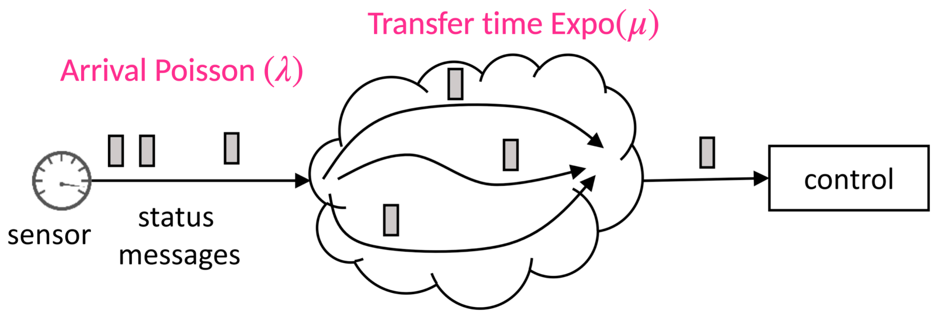
where $\boldsymbol{\Phi} = \theta \mathbf{I} + \mathbf{D} + \mathbf{M}$, $\boldsymbol{\Psi} = \boldsymbol{\Phi}^{-1} \boldsymbol{\Lambda}$ and matrices have $I_{\max} + 1$ lines and columns

⇒We obtain $\tilde{f}_A(\theta) = \mathbb{E}^0 \left(e^{-\theta X_0} \right)$ as a rational fraction in θ

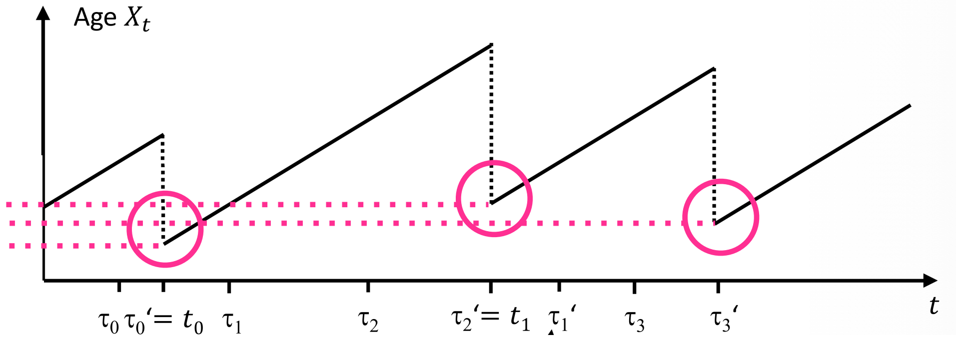
⇒We obtain pdf and CDF as combination of exponentials and polynomials, with explicitly computed coefficients.

Recap

We have obtained a method to compute the distribution of age of information at instants of arrival of informative messages.



We want the distribution of age of information at an arbitrary point in time.



The Inversion Formula of Palm Calculus

Assume: the quantity Y_t is jointly stationary with the underlying system state Z_t hence with the point process T_n, of intensity λ; then:

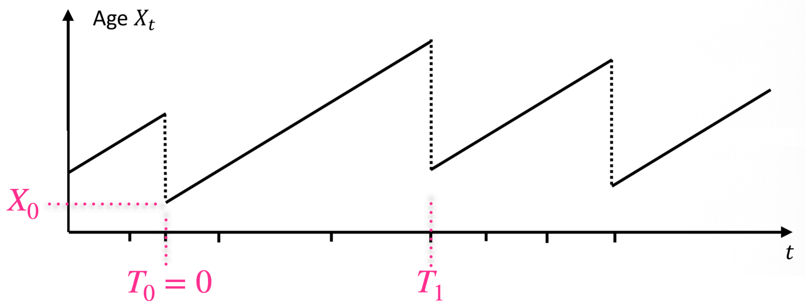
Theorem: $\mathbb{E}(Y_t) = \mathbb{E}(Y_0) = \lambda \mathbb{E}^0 \left(\int_0^{T_1} Y_s ds \right)$

Says that the expectation of Y_t, at an arbitrary point in time, is equal to $\lambda \times$ the expectation, at an arbitrary event, of the integral of Y_t between two events.

Example: $Y_t = 1$ (intensity formula):

$$\frac{1}{\lambda} = \mathbb{E}^0 \left(T_1 \right) = \mathbb{E}^0 \left(T_1 - T_0 \right)$$

Age of Information at Arbitrary Point in Time



Apply inversion formula to $Y_t = \varphi(X_t)$ where φ is any test function:

$$\mathbb{E}\left(\varphi(X_t)\right) = \hat{\lambda}\mathbb{E}^0\left(\int_0^{T_1} \varphi(X_s)ds\right) = \hat{\lambda}\mathbb{E}^0\left(\int_0^{T_1} \left(\varphi(X_0 + s)\right) ds\right)$$

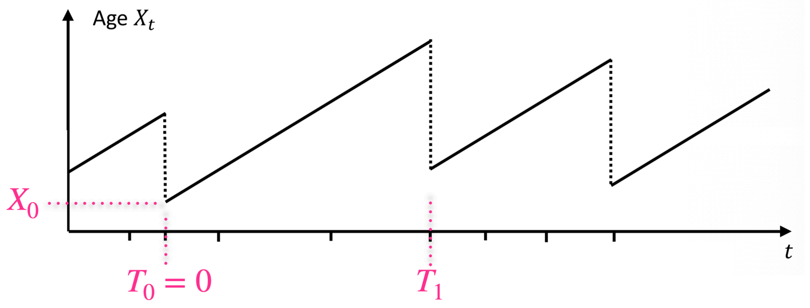
Let $f^0(x_0, t_1)$ be the joint *Palm* PDF of (X_0, T_1), i.e. the joint pdf of (age, time until next informative arrival) observed at an informative arrival; thus

$$\mathbb{E}\left(\varphi(X_t)\right) = \hat{\lambda}\int_0^{+\infty}\int_0^{+\infty}\int_0^{t_1} \varphi(x_0 + s)f^0(x_0, t_1)dsdx_0dt_1 = \int_0^{+\infty}\varphi(u)\int_0^{u}\int_{u-x_0}^{+\infty} f^0(x_0, t_1)dt_1dx_0du$$

Now $\mathbb{E}\left(\varphi(X_t)\right) = \int_0^{+\infty} \varphi(x)f(x)dx$ where $f(x)$ is the PDF of age of information at an arbitrary point in time. Thus:

$$f(x) = \hat{\lambda}\int_0^{x}\int_{x-x_0}^{+\infty} f^0(x_0, t_1)dt_1dx_0$$

Joint Palm PDF $f^0(x_0, t_1)$ of (X_0, T_1)



We know the probability $p^0_{n,n-i}$ that an arbitrary informative arrival is a transition $(n, n-i)$ of the Markov chain Z_t, hence

$$f^0(x_0, t_1) = \sum_{n,i} p^0_{n,n-i} f^0(x_0, t_1 \mid n, n-i)$$

where $f^0(x_0, t_1 \mid n, n-i)$ is the Palm PDF of (X_0, T_1) given that the informative arrival is a transition $(n, n-i)$.

Markov property $\Rightarrow$ future is independent of past given present state, thus:

$$f^0(x_0, t_1 \mid n, n-i) = f(x_0 \mid n, n-i) h(t_1 \mid n-i)$$

PDF of X_0 given transition is $(n, n-i)$

Its LST is $\tilde{f}_{n,n-i}(\theta)$, already computed

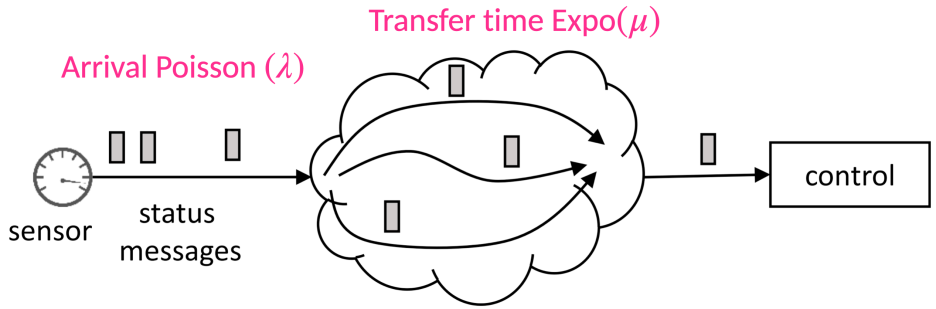
PDF of time from t until next informative arrival, given $Z_t = n - i$

Computed using Lemma 4

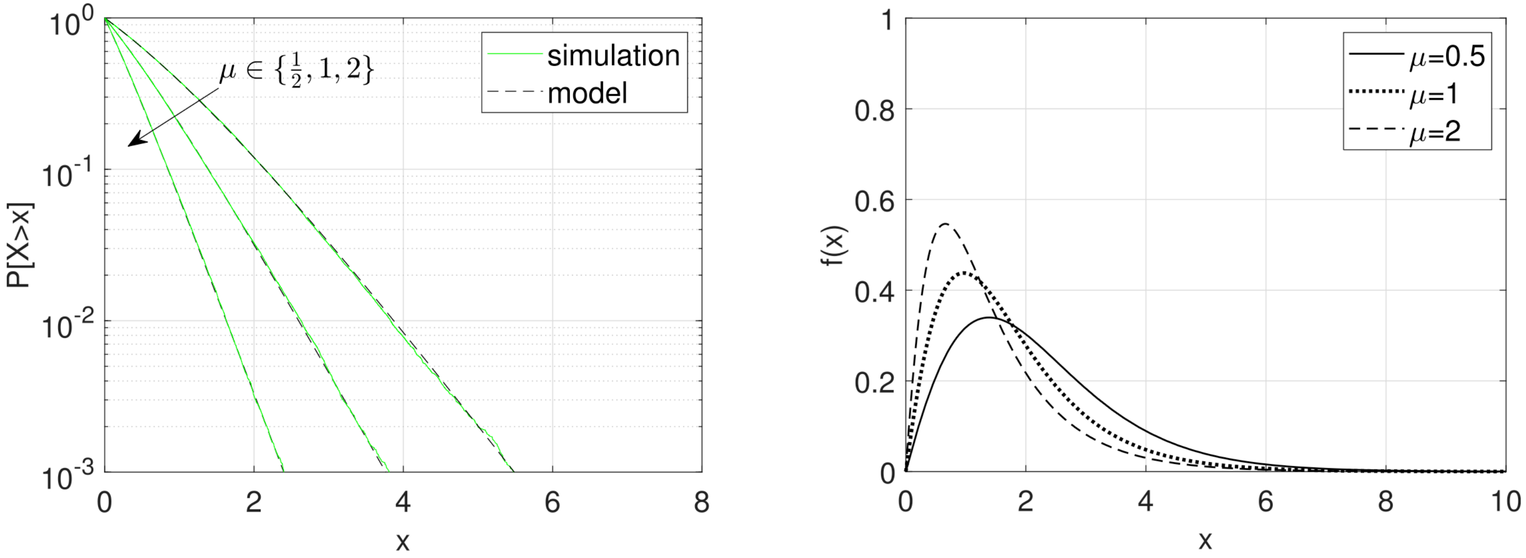
PDF $f(x)$ of Age of Information at an Arbitrary Point in Time

Theorem 5: In a stationary $M/M/I_{\max}/I^*_{\max}$ system, the PDF of the age of information, $f(x)$, is given by

$$f(x) = \hat{\lambda} \sum_{i=0}^{I_{\max}} \sum_{j=0}^{I_{\max}} \sum_{(n',n) \text{ s.t.} 1 \le n+1 \le n' \le I_{\max}} p^{\circ}_{n',n} \, e^{-x\tilde{d}_j} \times$$
$$\sum_{l=0}^{\tilde{\xi}_{j,n}+\xi_{i,n',n}} \tilde{c}_l(x) \left(\frac{l!}{(d'_i - \tilde{d}_j)^{l+1}} - e^{-x(d'_i - \tilde{d}_j)} \right.$$
$$\left. \sum_{v=0}^{l} \frac{l!}{v!(d'_i - \tilde{d}_j)^{l-v+1}} x^v \right)$$

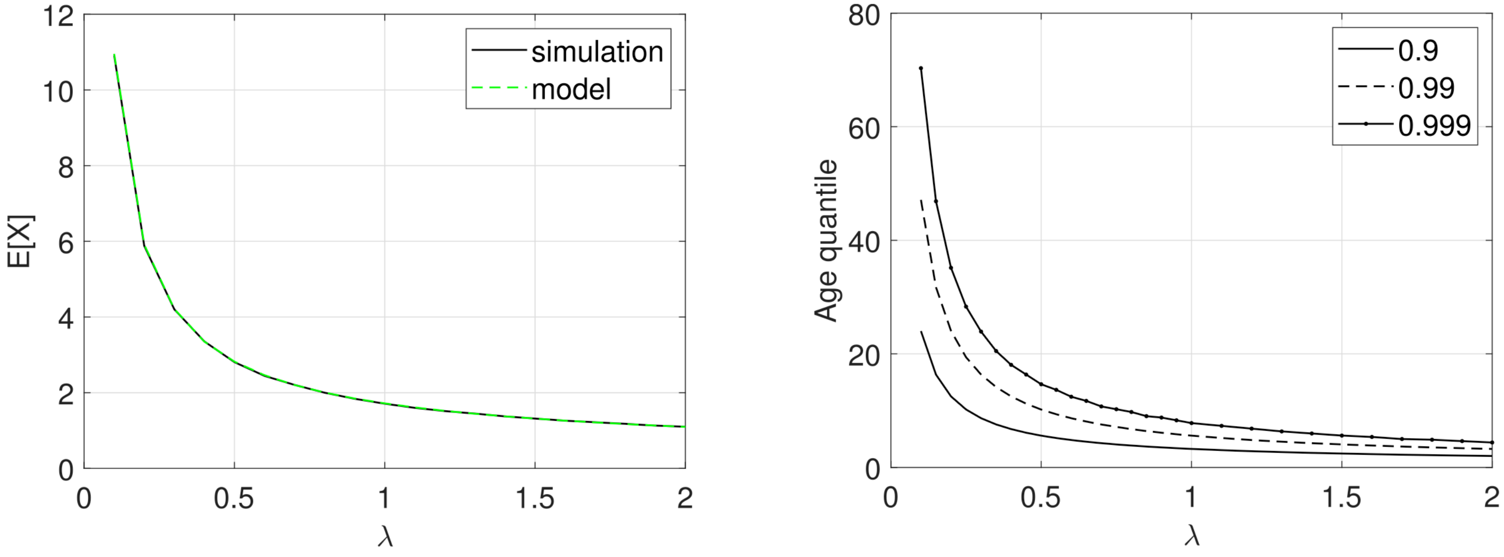


Here $\tilde{c}_l(x)$ is a polynomial in x, the coefficients of which are computed numerically for every (λ, μ). All other parameters are obtained in closed form.



(a) (b)

Fig. 6. (a) CCDF of the age at the arrival times of informative messages at the receiver for arrival rate $\lambda = 1$ and varying OWD parameter μ. $I_{\max} = 20$. (b) Probability density of the age $f(x)$ obtained from (9) for $\lambda = 1$ and varying OWD parameter μ. $I_{\max} = 20$.



(a) (b)

Fig. 7. (a) Expected age $E[X]$ obtained from simulations compared to the model in (8) with φ being the identity function for $I_{\max} = 20$ and $\mu = 1$. (b) Quantiles of the age $P[X > x_\varepsilon] = \varepsilon$ obtained from integrating the age density in (9) for $I_{\max} = 20$ and $\mu = 1$. The legend shows different values of $1 - \varepsilon$.

Conclusion

We have obtained the distribution of age of information for a simple model.

The method involves Palm calculus applied to finite state Markov chains.

Thank you !